THE NATURAL FREQUENCIES OF CANTILEVERED PLATES

by

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CHAPTER I

INTRODUCTION

It is on rare occasions that an engineering component or structure exists without being subjected to vibratory conditions during some part of its life. As it is not always possible or practical to carry our laboratory tests, it is essential for design engineers to be able to predict analytically, how structures and components will react to vibrating conditions. Exact theoretical analysis is possible for a wide range of vibrational problems and, for the solution of many others, approximate methods, which have proved to be very satisfactory, have been developed.

Among the problems for which approximate methods have been developed is that of the vibration of thin, flat, rectangular plates of uniform thickness. It is possible to derive a differential equation for a thin, flat plate of uniform thickness subjected to lateral vibrations, but it has been solved only for rectangular plates which are simply supported at all four edges and for rectangular plates which are simply supported at two opposite edges and having any conditions at the two other edges. For rectangular plates having any other boundary conditions, the solution of the differential equation is extremely complicated and it becomes necessary to use an approximate method to calculate their natural frequencies and corresponding modes of vibration.

One of the best known approximate methods of calculating the natural frequencies of thin, flat rectangular plates of uniform thickness and having any combination of fixed, simply supported or free edges is the Rayleigh-Ritz method and consists of equating the maximum potential energy and the maximum kinetic energy of the vibrating plate. The method was first introduced by Rayleigh (1894) and later by Ritz (1909) and was used by Ritz to calculate the natural frequencies, and the corresponding nodal patterns, of square plates with all edges free.

In recent years the method was used by Young (1950) and Warburton (1954) to calculate the natural frequencies of rectangular plates having various combinations of fixed, simply supported and free edges, and by Barton (1951) to calculate the natural frequencies of cantilevered, rectangular and parallelogram shaped plates. Young obtained the first six natural frequencies and the corresponding modes of vibration of a square plate fixed along two adjacent edges and free along the other two edges, of a cantilevered square plate and of a square plate fixed at all four edges. Warburton derived a simple approximate expression which can be used to calculate the natural frequencies of all modes of vibration of rectangular plates having any combination of free, simply supported and fixed edges. Barton obtained the first five natural frequencies and the corresponding nodal patterns of cantilevered rectangular plates having length to breadth ratios of $\frac{1}{2}$, 1, 2 and 5 and the first two natural frequencies and the corresponding modes of vibration of cantilevered parallelogram shaped plates having sides of equal length and skew angles of 15°, 30° and 45°.

With the advent of high speed aircraft and missiles, it has become necessary to extend the work on the vibration of thin, flat plates to include cantilevered, triangular and trapezoidal shaped plates, as they are first approximations to the shape of the wings and fins of the aircraft and missiles.

Gustafson, Stokey and Zorowski (1953) obtained experimentally the first six natural frequencies and the corresponding nodal patterns of five cantilevered, right-angled, triangular plates having length to fixed edge ratio varying from two to twenty, and of five cantilevered, triangular plates having constant length to fixed edge ratios with sweep back angles varying from 0° to 45°. In a further series of tests (1954) they investigated, experimentally, the effect on the natural frequencies of removing increasing amounts from the tip of cantilevered, triangular plates. They obtained the first six natural frequencies and the corresponding nodal patterns of three sets of six plates. Two of the sets originated from right-angled triangular plates and the other from a triangular plate of swept back plan form. They have also described methods, (1953 and 1954) which depend on known experimental frequencies of calculating the natural frequencies of plates differing in plan form from those investigated experimentally. For their experimental work, the fixed edge of the plate was obtained by clamping it, at the base of the triangle, between two steel blocks. Heiba (1954), who obtained experimentally the first six natural frequencies of a series of cantilevered rectangular and trapezoidal plates, also obtained the fixed edge of the plate by clamping it at one end. He, however, expressed doubts about the effectiveness of that method of obtaining a fixed edge. As a result the experimental results, which are presented in this report were obtained from test specimens which originated from thick steel blocks which were machined over a portion of their length to make a thin flat plate of uniform thickness.

Cox and Klein, two American investigators, have published several papers, in which the method of collocation is used to calculate the fundamental natural frequency of triangular and trapezoidal shaped plates, having various combinations of fixed and simply supported edges.

Kawashima (1958) used the finite-difference method to calculate the first five natural frequencies of a cantilevered, right-angled, triangular plate, having a length to fixed edge ratio of one, and the first three natural frequencies of a cantilevered, right-angled, triangular plate, having a length to fixed edge ratio of a half.

Stanisic and McKinley (1961) used the Galerkin method to calculate the fundamental natural frequency of symmetrical trapezoidal plates with all edges fixed.

Andersen (1954) used the Rayleigh-Ritz method to calculate the natural frequencies of the first two symmetric and the first two antisymmetric modes of vibration of four cantilevered, isosceles triangular plates, having length to fixed edge ratios of one, two, four and seven. He also calculated the natural frequencies of the first two symmetric modes of vibration of three cantilevered, right-angled, triangular plates having length to fixed edge ratios of two, four and seven. The results for the symmetric modes were compared with results obtained for a cantilevered, triangular beam of uniform thickness, and it was found that the agreement was good. Agreement between the calculated frequencies of the symmetrical modes and the experimental frequencies of Gustafson, Stokey and Zerowski was also good. He does not, however, compare the calculated frequencies of the antisymmetric modes of the isosceles triangular plates with other results, and on comparing them with the experimental results in this report Andersen's values for the antisymmetric modes tend to be high.

Andersen assumed the deflected form of the vibrating plate to be a series consisting of the appropriate functions which represent the normal modes of vibration of uniform beams. In the calculations a co-ordinate transformation was used to give constant limits of integration over the area of the plate. The co-ordinate transformation altered the expressions for the maximum potential energy and the maximum kinetic energy of the vibrating plate, and made it necessary to modify slightly the beam functions which were used to represent the deflected form of the vibrating plate. Because of these alterations the functions to be integrated in the expressions for the maximum potential energy and the maximum kinetic energy, are extremely complicated. It has been found, however, as a result of the present investigations that the problem of calculating the natural frequencies of triangular plates by the Rayleigh-Ritz method can be simplified, by avoiding the co-ordinate transformation used by Andersen.

Although the problem of the vibration of thin, flat triangular and trapezoidal shaped plates has been tackled both from the experimental and the theoretical point of view, there remains a great deal of work to be done in this field. Most of the authors who have investigated the problem theoretically have confined their investigations to the fundamental or the first few natural frequencies of these plate shapes. The object of the present research programme is to try to obtain a method which can be used to calculate the natural frequencies of all modes of vibration of thin, flat, cantilevered, triangular and trapezoidal shaped plates and, if successful, to investigate the possibility of extending it to include triangular and trapezoidal shaped plates having other combinations of boundary conditions.

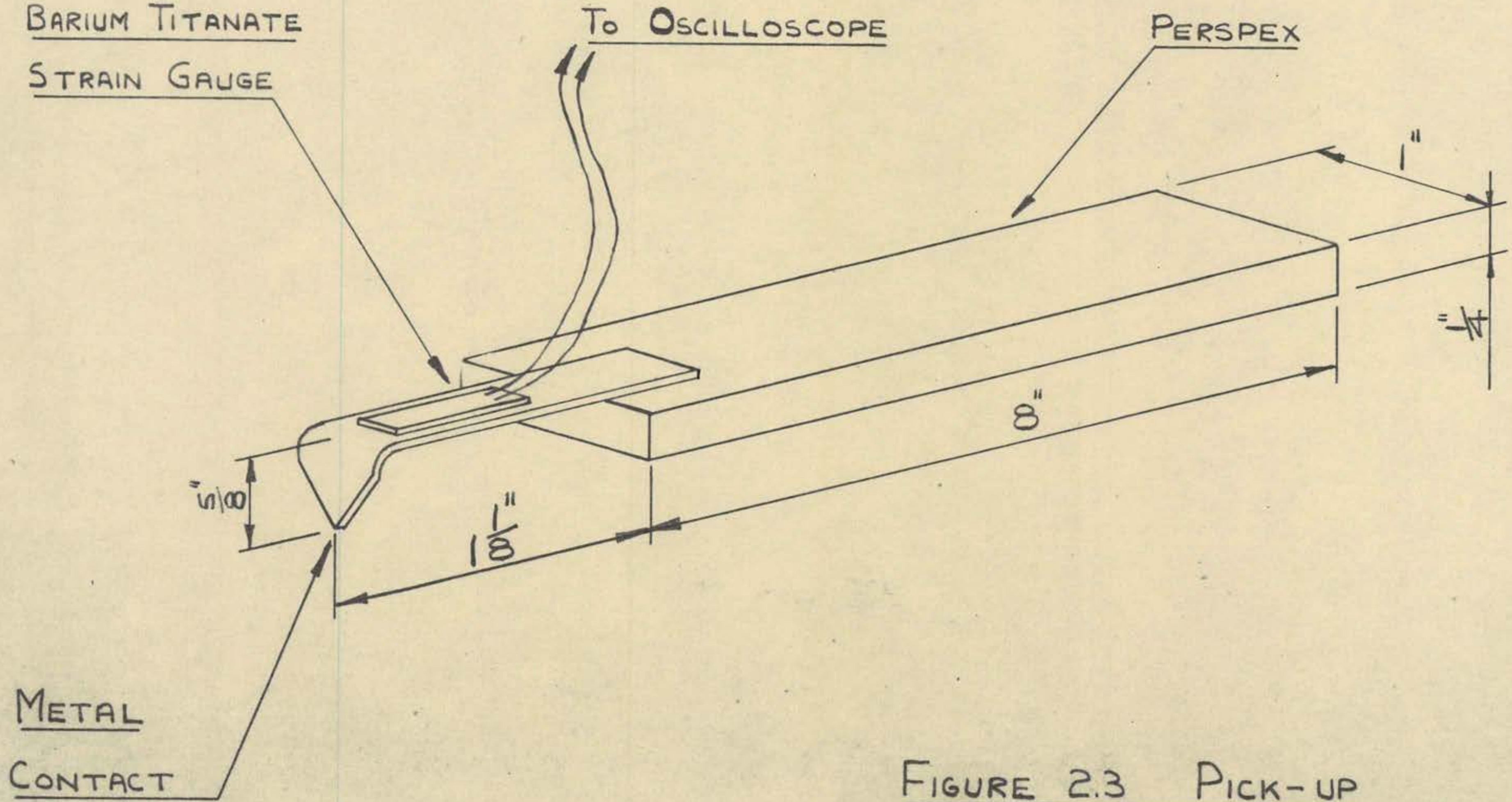


FIGURE 2.3 PICK-UP

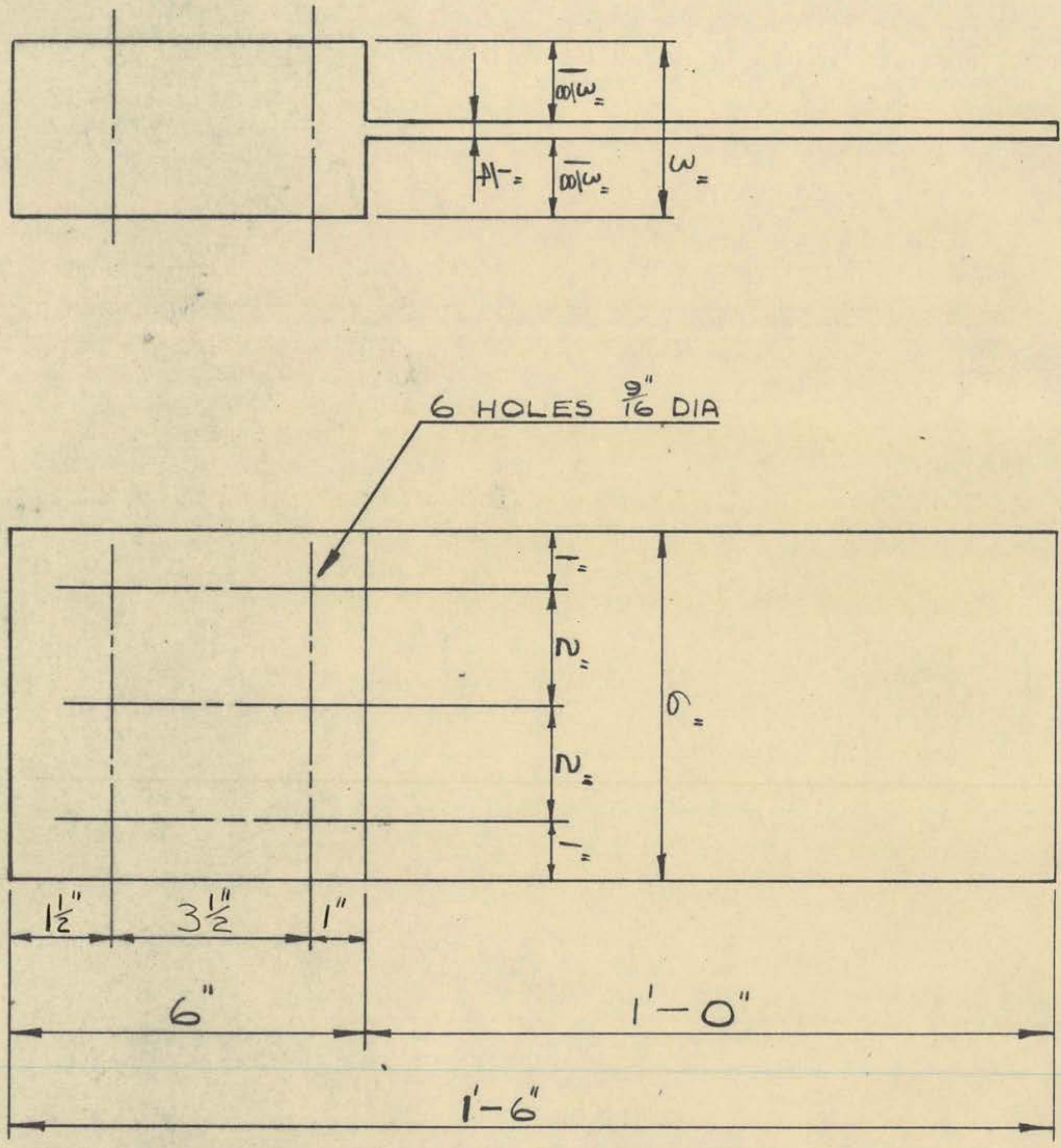


FIGURE 2.2 TEST SPECIMEN

SCALE : 3/8" ≡ 1"

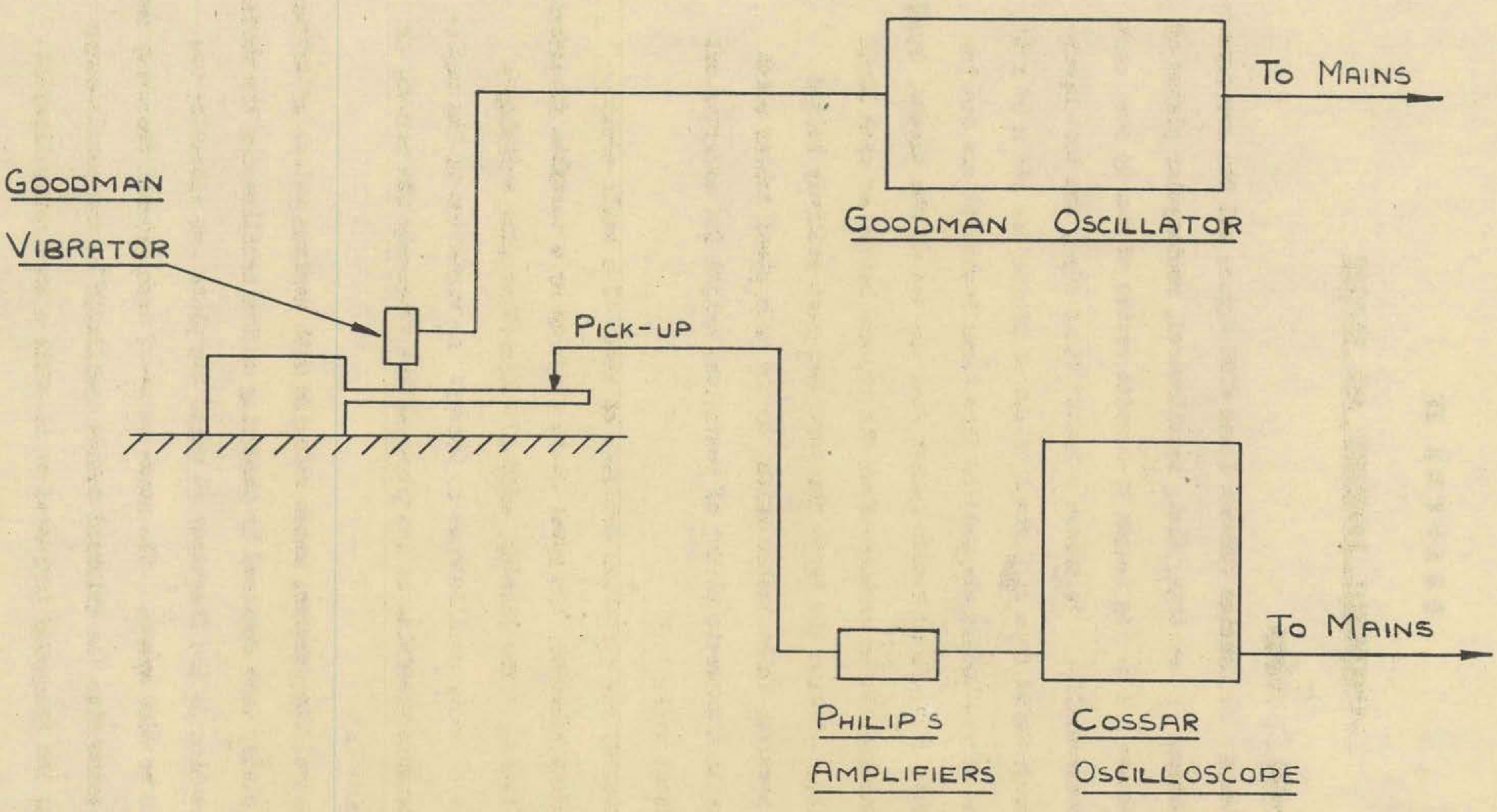


Figure 2.1 Diagram of Experimental Equipment

CHAPTER II

EXPERIMENTAL PROCEDURE AND RESULTS

2.1 Experimental Procedure

The natural frequencies between 0 and 8000 c.p.s. and the corresponding nodal patterns of two thin, flat, cantilevered, rectangular plates of uniform thickness and having length to breadth ratios of two to one, were obtained experimentally. To obtain a "good" fixed edge, two test specimens were manufactured from mild steel blocks of dimensions 18" x 6" x 3". The thickness of the blocks was reduced from three inches to one quarter of an inch over a length of twelve inches from one end of the blocks, equal amounts of material being machined from the top and bottom of each block (see Fig. 2.2). During the tests the specimens were securely bolted through the portion, three inches thick, to two mild steel joists which were built in to a concrete plinth of convenient height for carrying out the experimental work.

Vibration of the plate was achieved by means of a small electromagnetic Goodman vibrator, the power being supplied by a variable frequency Goodman oscillator. The pick-up, used in conjunction with a Philip's amplifier and a Cossor oscilloscope to measure the variation of the amplitude of transverse vibration of the plate as the frequency was varied, is shown in Figure 2.3.

The natural frequencies, which coincide with maximum values of deflection of the plate, were obtained by observing on the oscilloscope the variation of deflection as the frequency at which the plate was vibrating was varied from 0 to 8000 ~/sec. The exact value of each natural frequency was obtained by connecting the Muirhead decade oscillator to the oscilloscope and adjusting the frequency indicated on it until a stationary Lisajons corresponding to many of the natural frequencies do not take the form of lines approximately parallel and perpendicular to the fixed edge of the plate. To determine the nodal patterns of these frequencies the pick-up was moved along each of the lines drawn parallel and perpendicular to the fixed edge of the plate, and the points of zero deflection on each line were marked. The shape of the nodal pattern was then determined by using the pick-up to determine the shape of the lines which joined the points of zero deflection.

At several of the natural frequenices of some of the plates investigated an attempt was made to obtain the exact shape of the nodal patterns by sprinkling sand on the vibrating plate. In each case, the only nodal pattern to be outlined was the one which had one nodal line parallel and one nodal perpendicular to the fixed edge of the plate. Because of the failure of the sand to outline the nodal patterns corresponding to the natural frequencies, the method described above was used.

2.2 Nodal Patterns

In general the nodal patterns corresponding to the natural frequencies of rectangular plates having any combination of boundary conditions, take the form of lines which are approximately parallel to the sides of the plate. Cantilevered, rectangular plates are not an exception and their nodal patterns can be used to divide the natural frequencies of the plate into families. The modes of vibration of the frequencies in any particular family have the same number of nodal lines perpendicular to the fixed edge of the plate.

For the range of frequencies investigated it was found that for the rectangular plate natural frequencies of the families with no, one, two and three nodal lines perpendicular to the fixed edge were included. As the area of the plate was reduced, the value of the natural frequency of each mode increased. As a result the number of natural frequencies which were in the range of frequencies investigated, decreased as the area of the plate was reduced.

With only a slight reduction in the area of the plate from a rectangle, the nodal patterns corresponding to the natural frequencies of the particular trapezium under consideration still take the form of lines which are approximately parallel and perpendicular to the fixed edge of the plate. As the shape of the plate approached a triangle, the nodal patterns do not take the form of lines which are approximately parallel and perpendicular to the fixed edge. It is still possible, however, despite the peculiar nodal patterns, to group the frequencies into families.

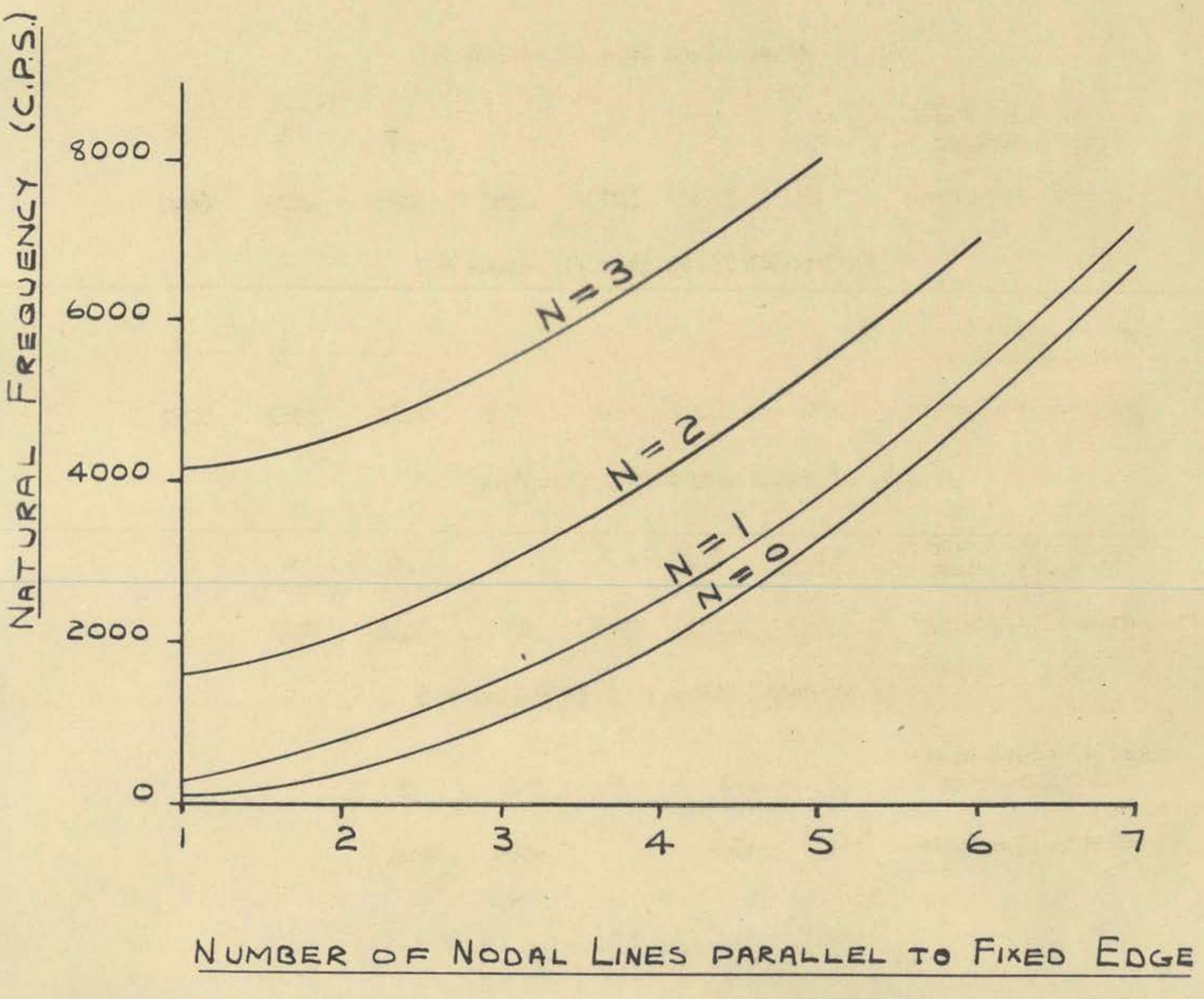
The exact shape of the nodal patterns of the first few natural frequencies in each family were obtained in the manner described in the previous section for the plate shapes which had nodal patterns with lines parallel and perpendicular to the fixed edge. When the shape of the plates were such that the nodal patterns did not take the form of lines parallel and perpendicular to the fixed edge, the exact shape of all the nodal patterns were obtained.

The nodal patterns which were recorded are shown in the next section and it can be seen that the drift from being parallel and perpendicular to the fixed edge starts when the reduction in area of the plate is approximately half way towards a triangle.

NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·25"

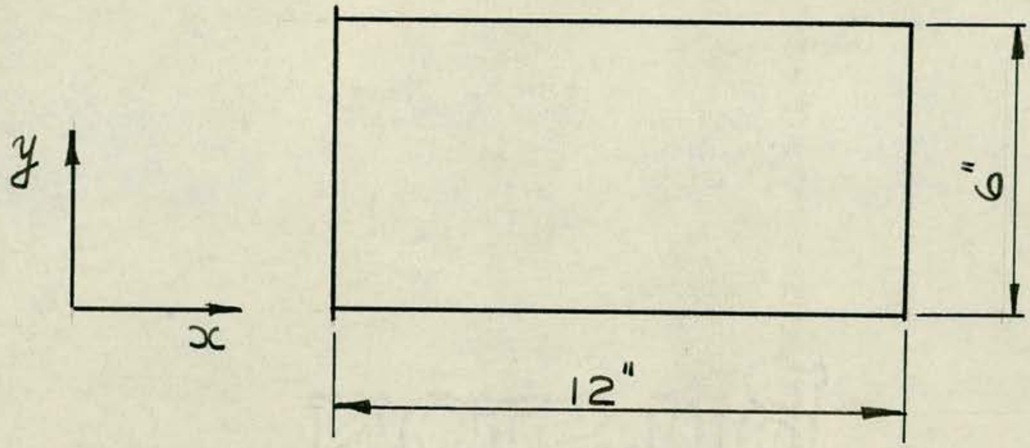
N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE



2.3 Experimental Results

Natural Frequencies and corresponding Modes of Vibration of Cantilevered Symmetrical Plates.

(a) Rectangular Plate



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	58	359	1003	1970	3255	4823	6690

No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	256	820	1558	2545	3808	5353	7251

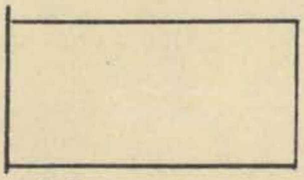
No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3	4	5	6
Natural Frequency	1561	2111	2963	4082	5435	7041

No. of nodal lines in y direction = 3

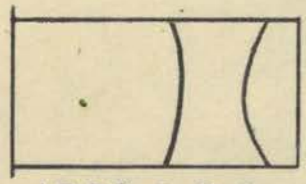
No. of nodal lines in x direction	1	2	3	4	5
Natural Frequency	4143	4591		6599	8024

Shape of Nodal Patterns Corresponding to Natural Frequencies

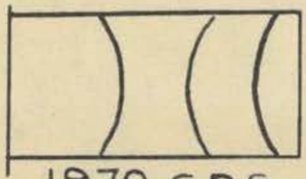


58 C.P.S.

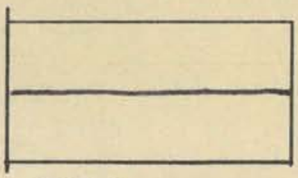
359 C.P.S.



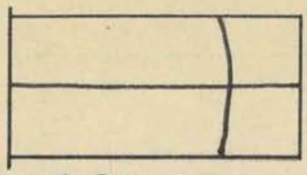
1003 C.P.S.



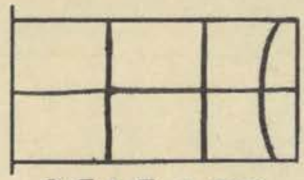
1970 C.P.S.



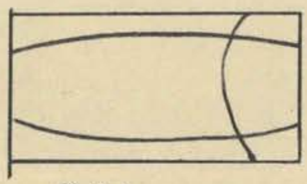
256 C.P.S.



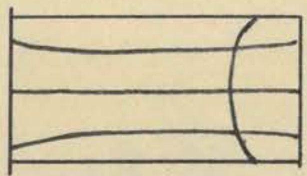
820 C.P.S.



2545 C.P.S.



2111 C.P.S.

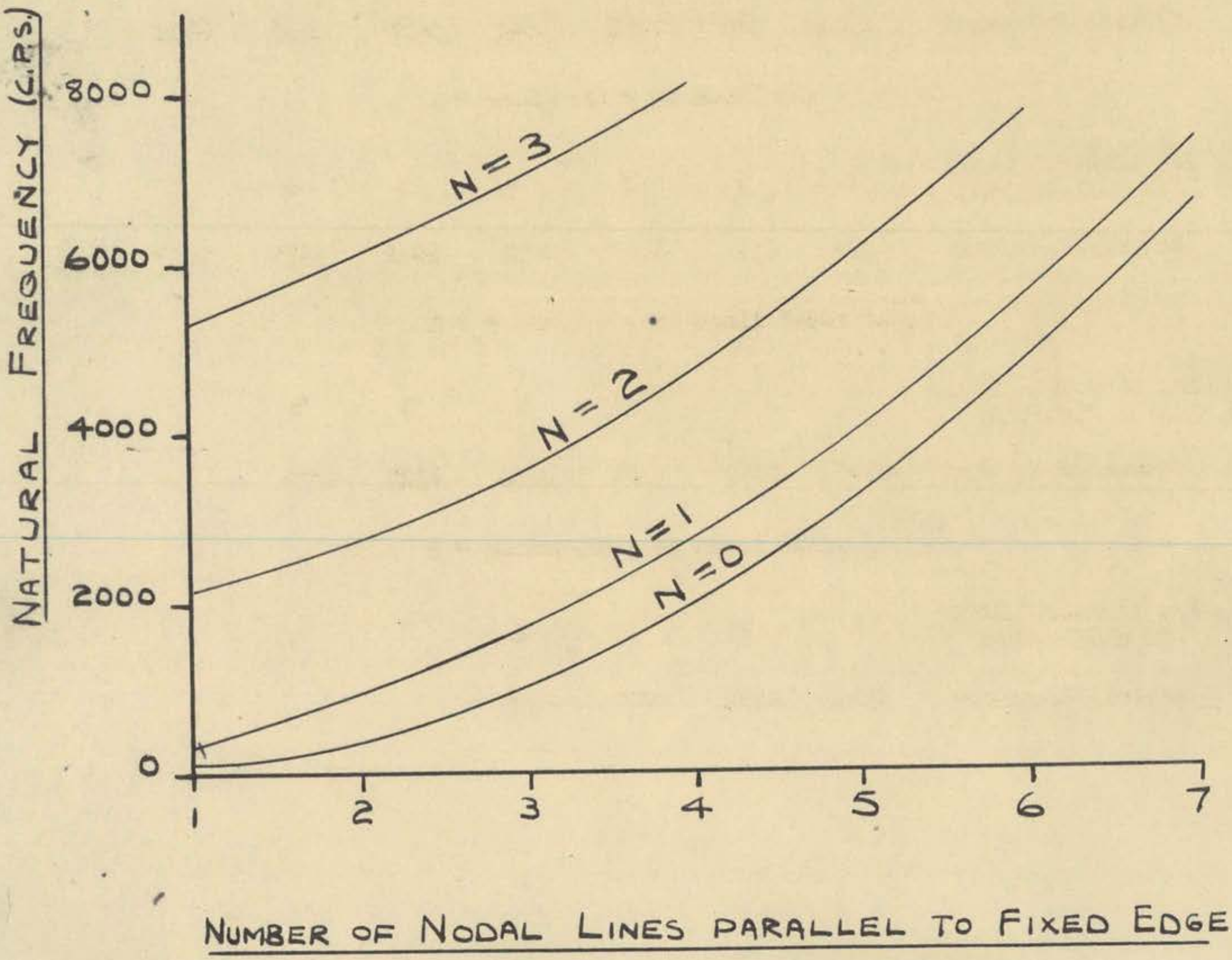


4591 C.P.S.

NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

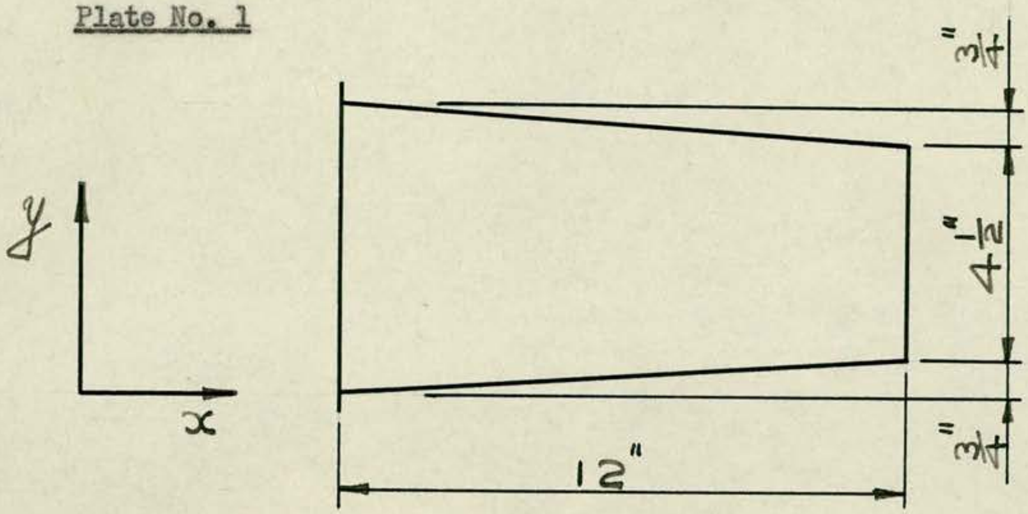
THICKNESS OF PLATE = 0·25"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE



(b) Symmetrical Trapezoidal Plates

Plate No. 1



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	63.4	369	1011	1966	3237	4846	6683

No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	322	939	1728	2739	4016	5577	7417

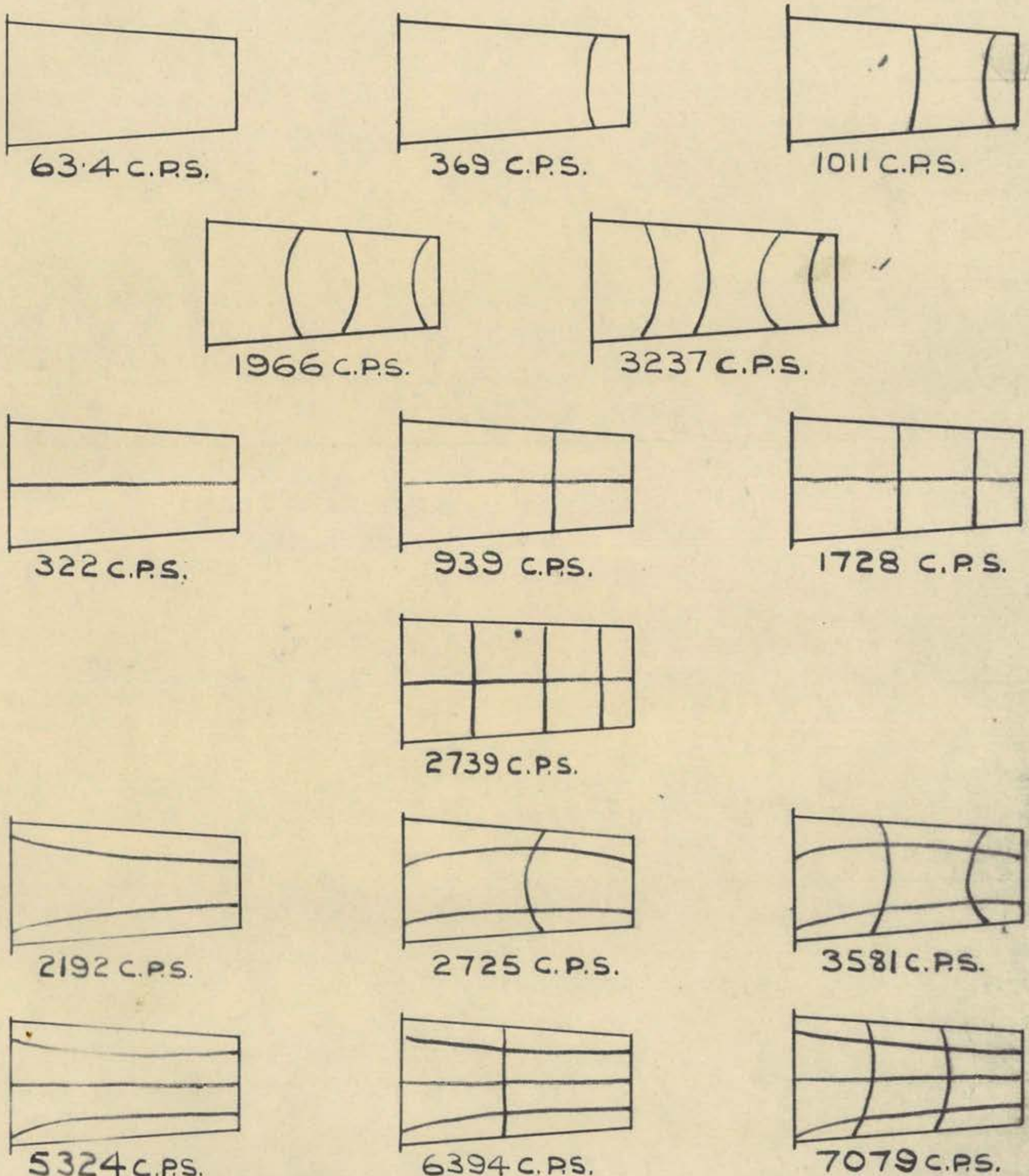
No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3	4	5	6
Natural Frequency	2192	2725	3581	4706	6127	7773

No. of nodal lines in y direction = 3

No. of nodal lines in x direction	1	2	3	4
Natural Frequency	5324	6394	7079	8133

SHAPE OF NODAL PATTERNS CORRESPONDING TO NATURAL FREQUENCIES



NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·25"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE

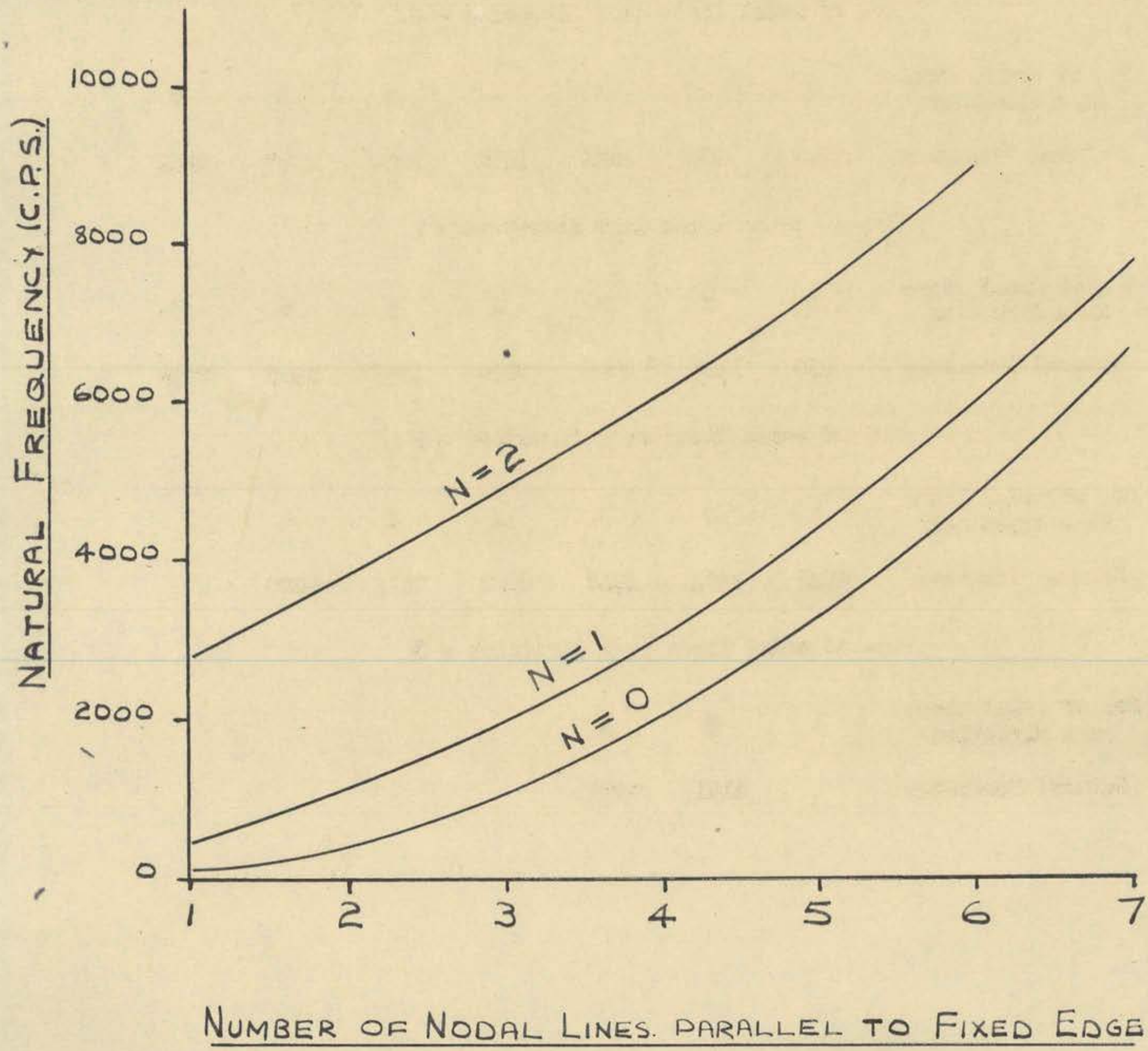
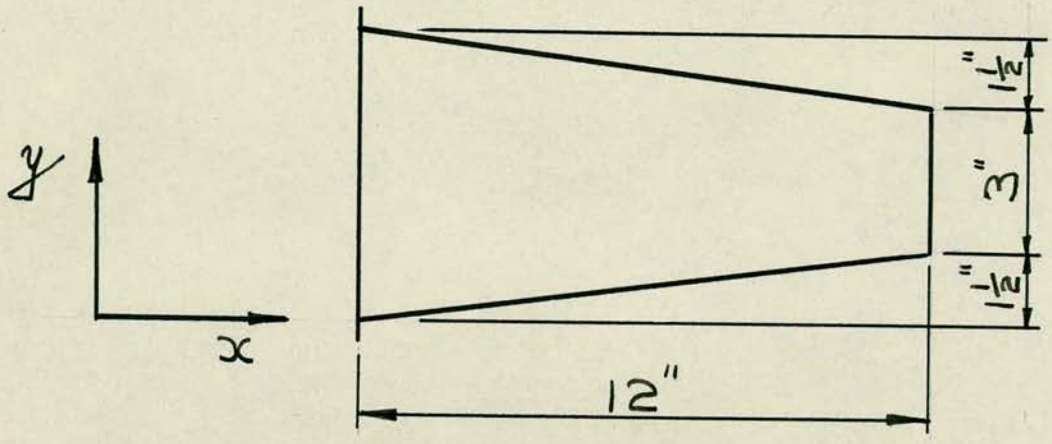


Plate No. 2



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	70.2	380	1022	1976	3233	4757	6668

No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	430	1119	1994	3079	4395	5968	7776

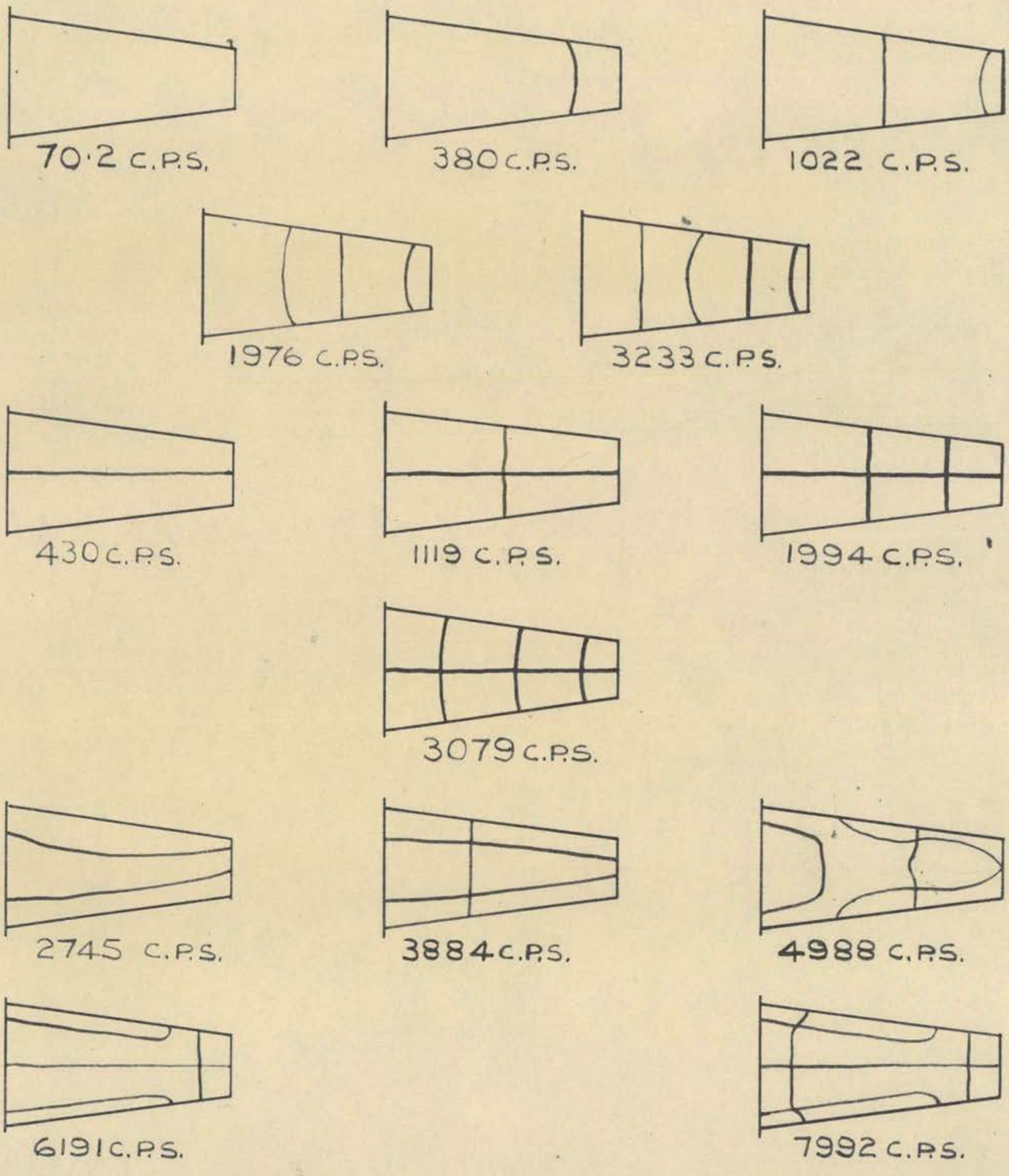
No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3	4	5	6
Natural Frequency	2745	3884	4988	6022	7414	8987

No. of nodal lines in y direction = 3

No. of nodal lines in x direction	1	2	3
Natural Frequency		6191	7992

Shape of Nodal Patterns corresponding to Natural Frequencies



NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·25"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE

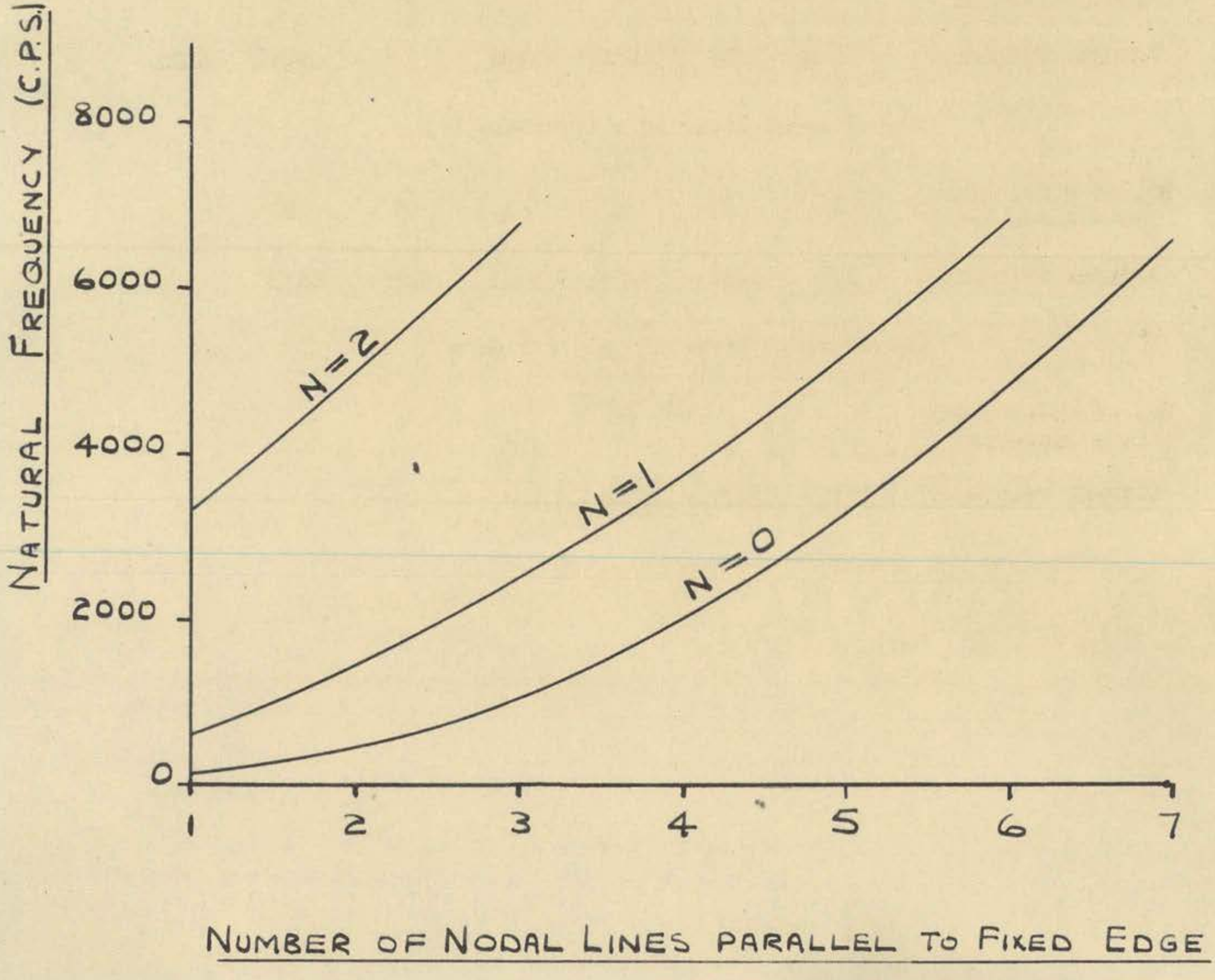
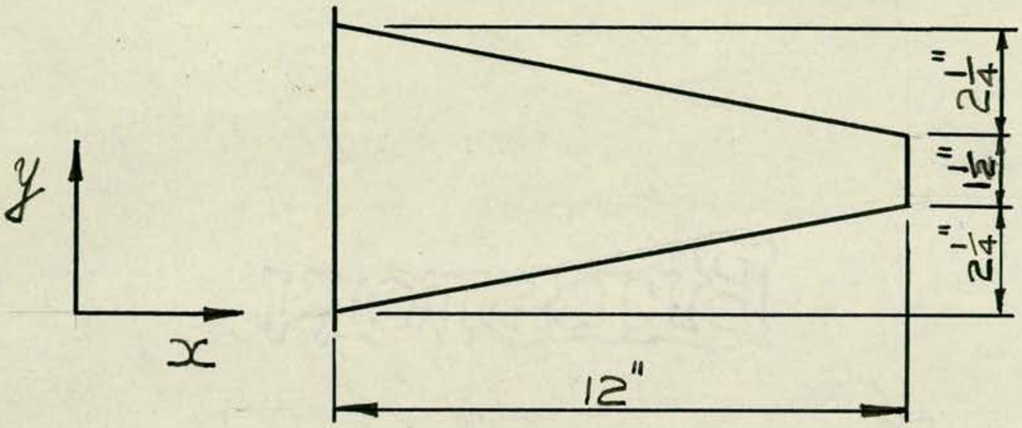


Plate No. 3



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	83.4	409	1046	1996		4792	6604

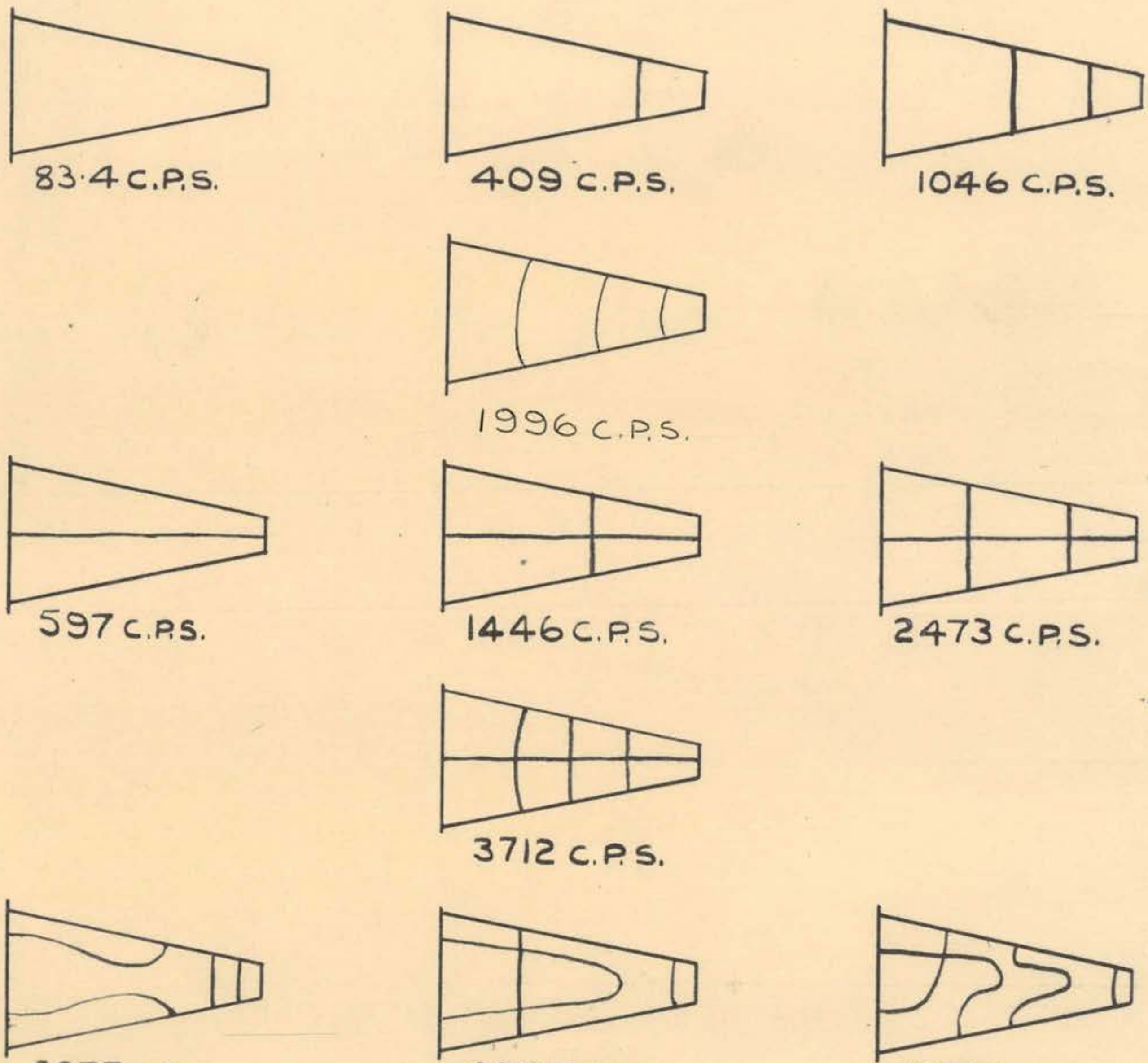
No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6
Natural Frequency	597	1446	2473	3712	5175	6833

No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3
Natural Frequency	3277	4987	6780

SHAPE OF NODAL PATTERNS CORRESPONDING TO NATURAL FREQUENCIES



NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·25"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE

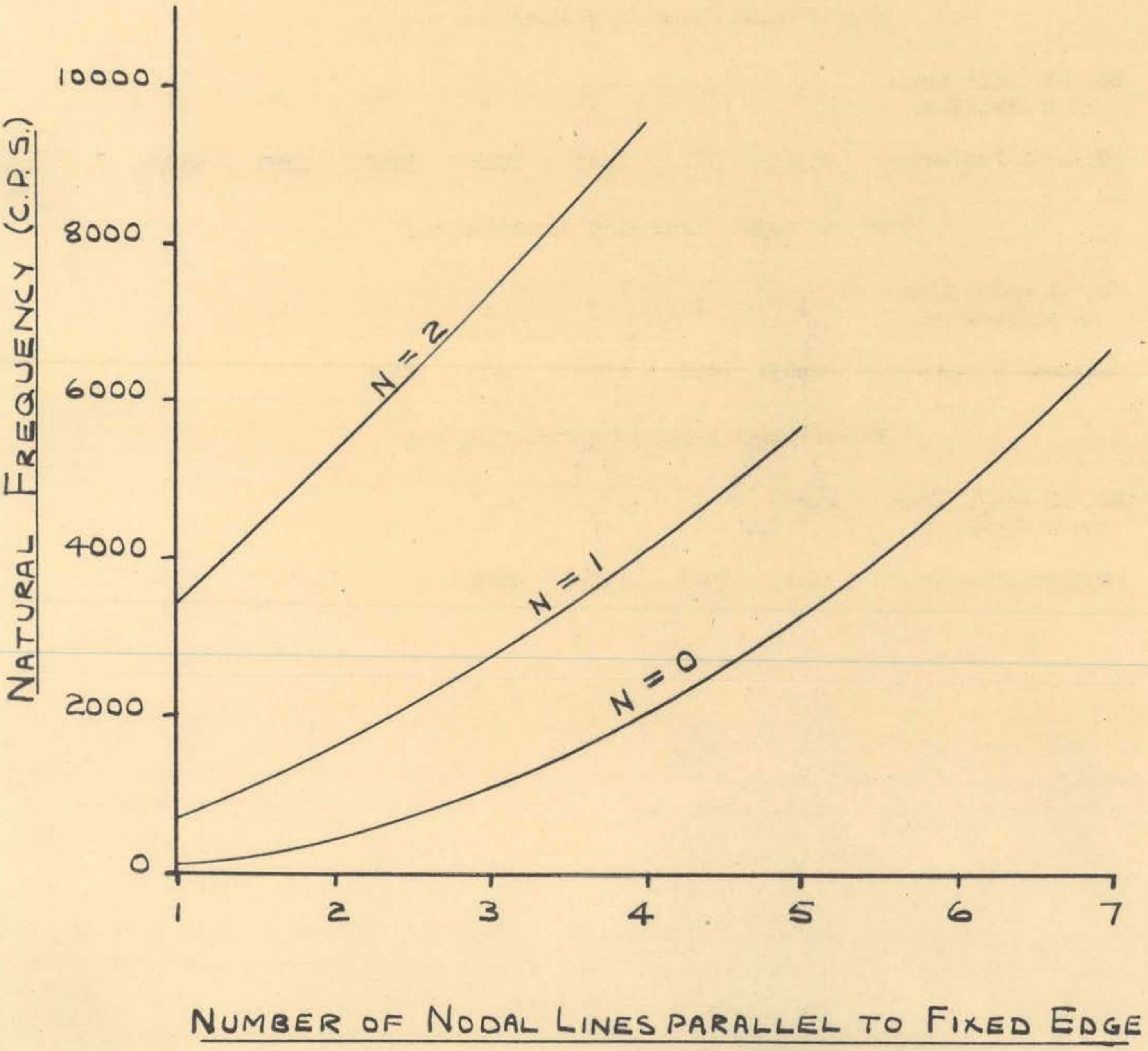
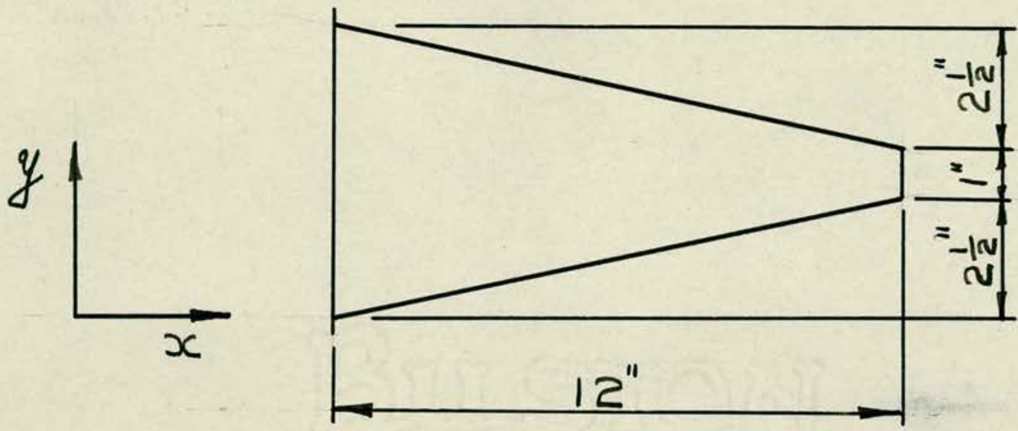


Plate No. 4



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	90.1	422	1062	2011	3250	4807	6632

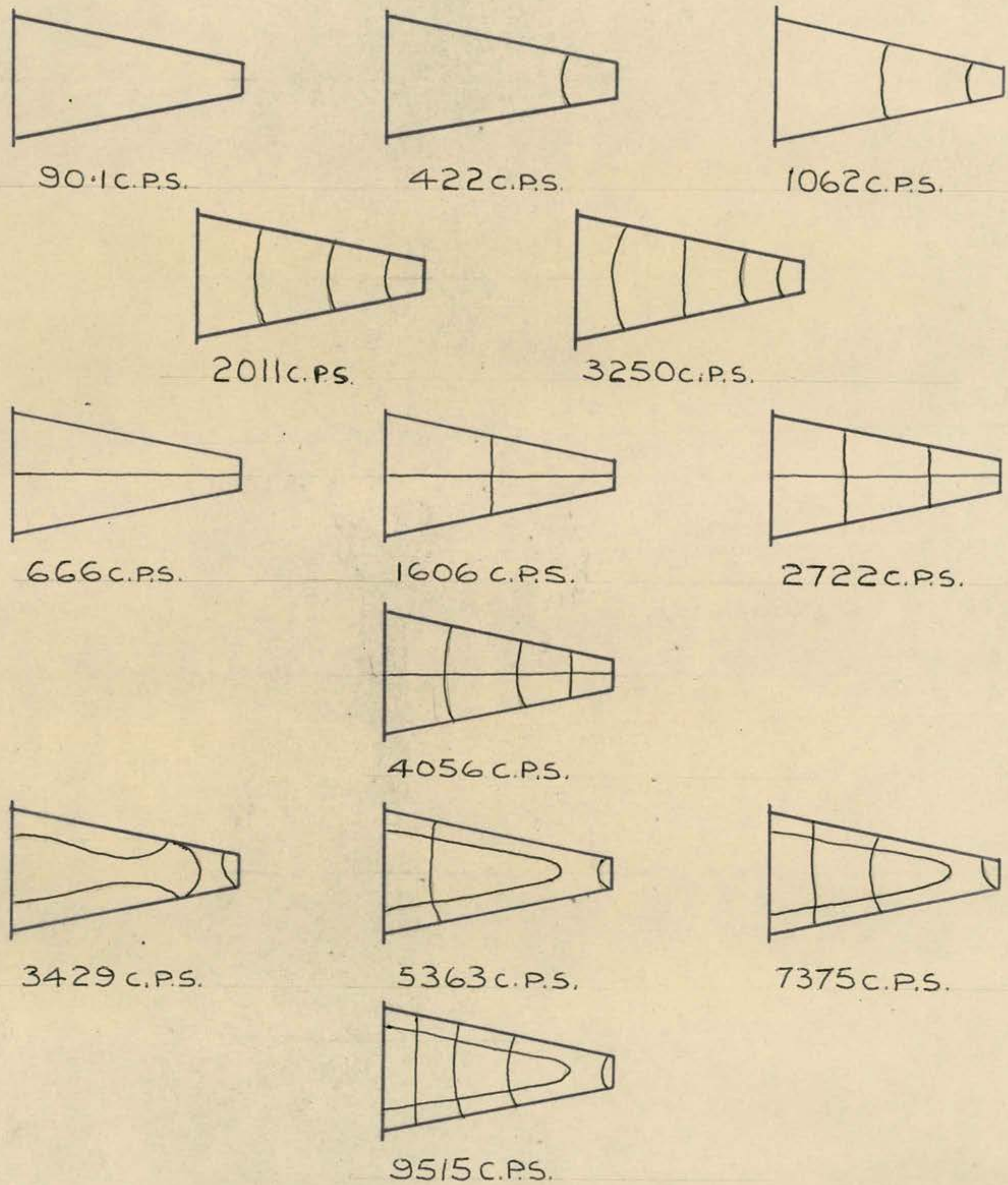
No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5
Natural Frequency	666	1606	2722	4056	5607

No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3	4
Natural Frequency	3429	5363	7375	9515

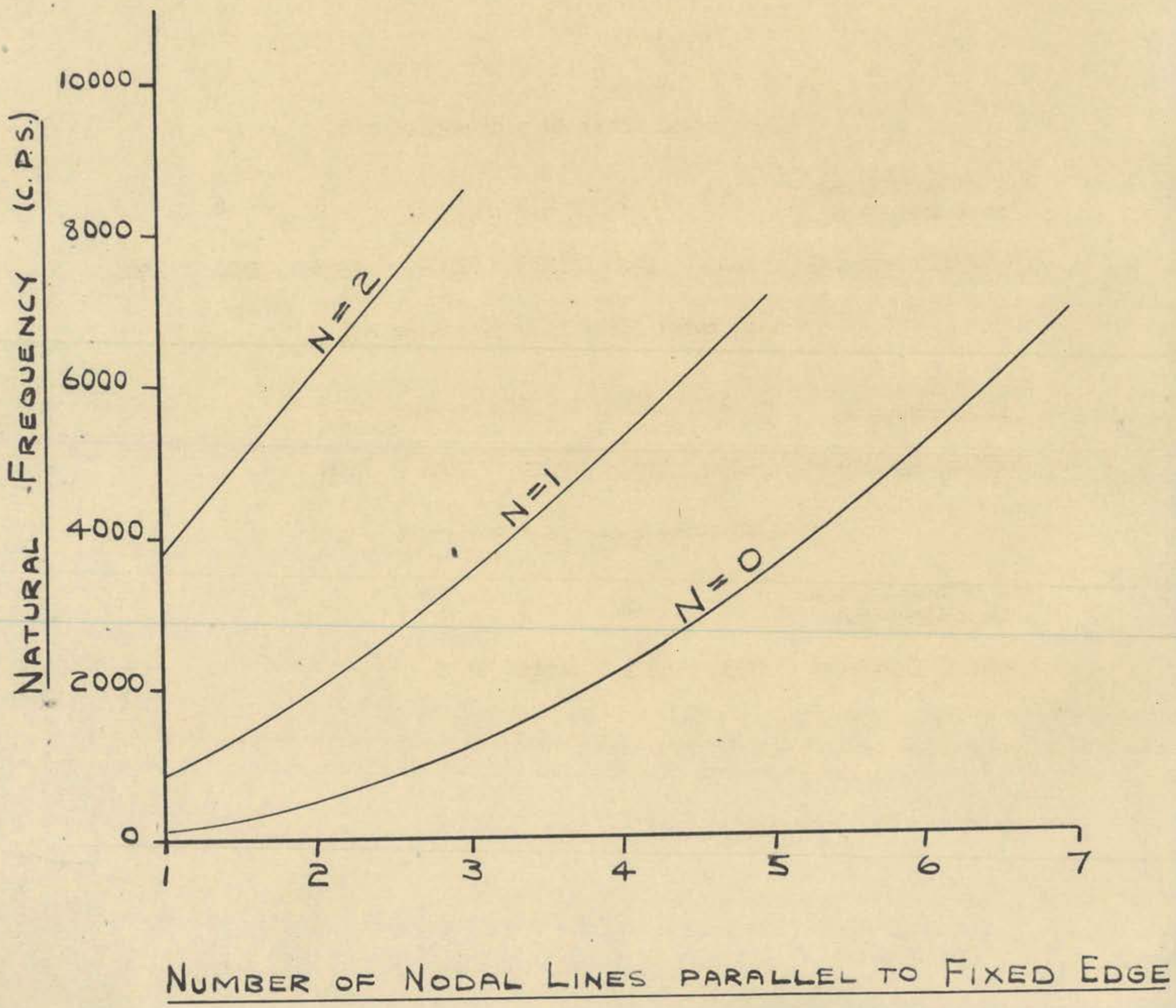
Shape of Nodal Patterns corresponding to Natural Frequencies



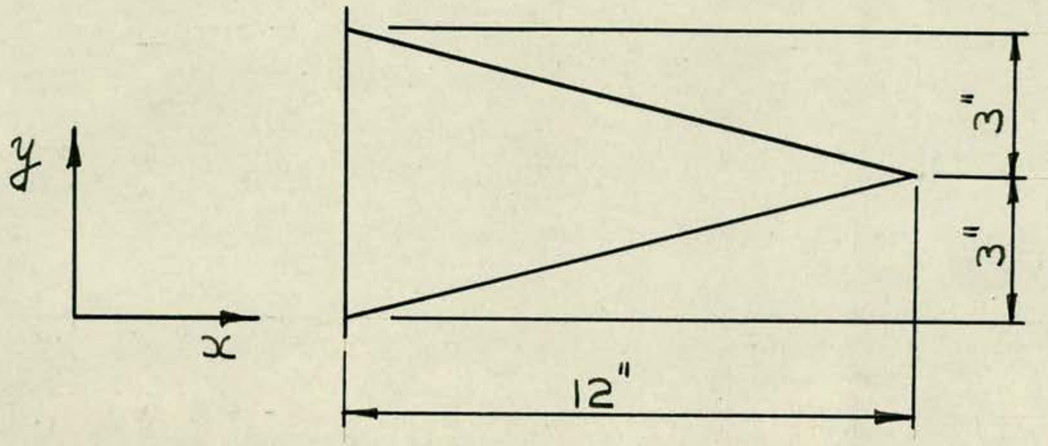
NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·25"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE



(c) Isosceles Triangular Plate



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	114	494	1193	2202	3485	5100	6964

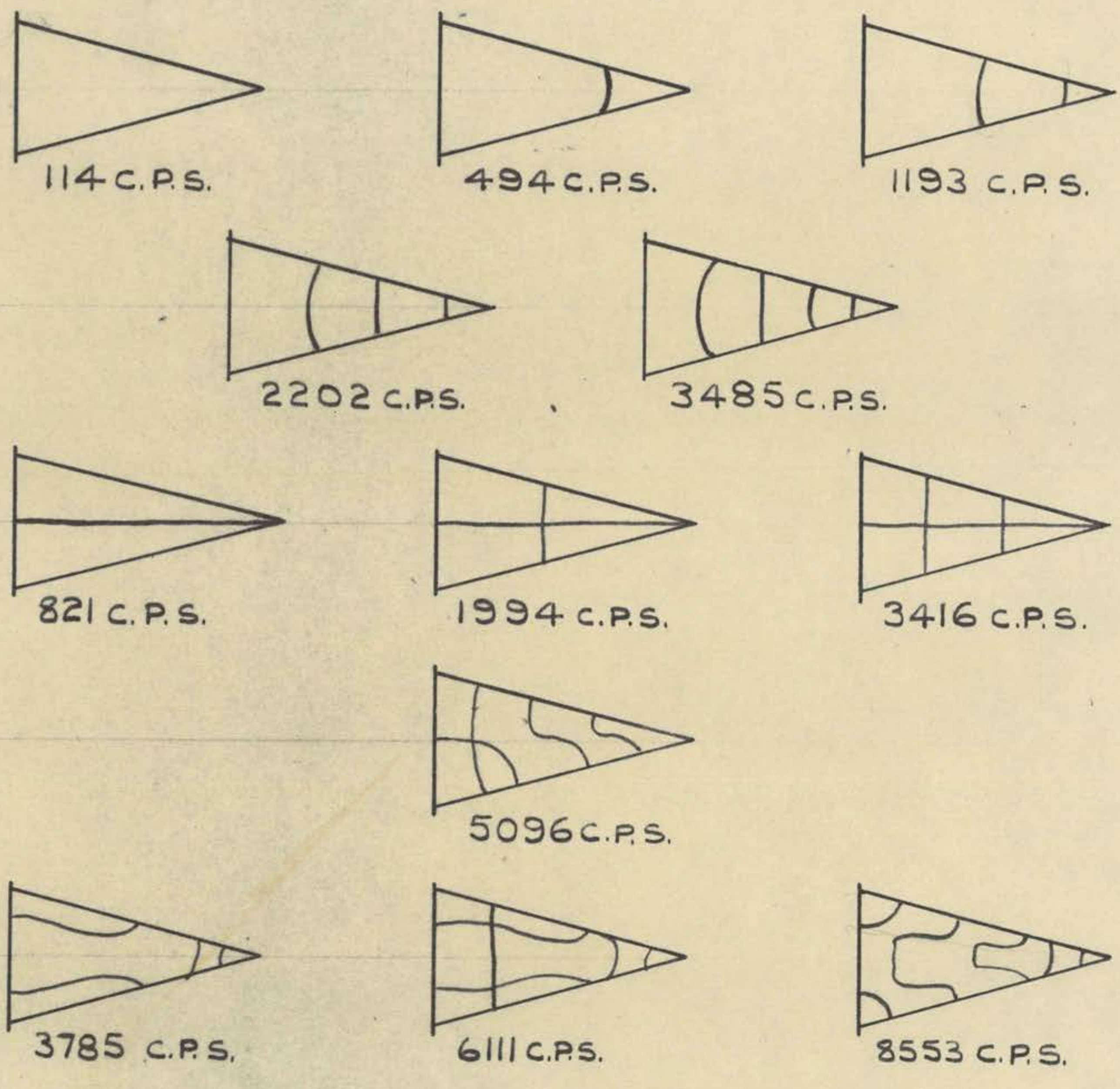
No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5
Natural Frequency	821	1994	3416	5096	7009

No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3
Natural Frequency	3785	6111	8553

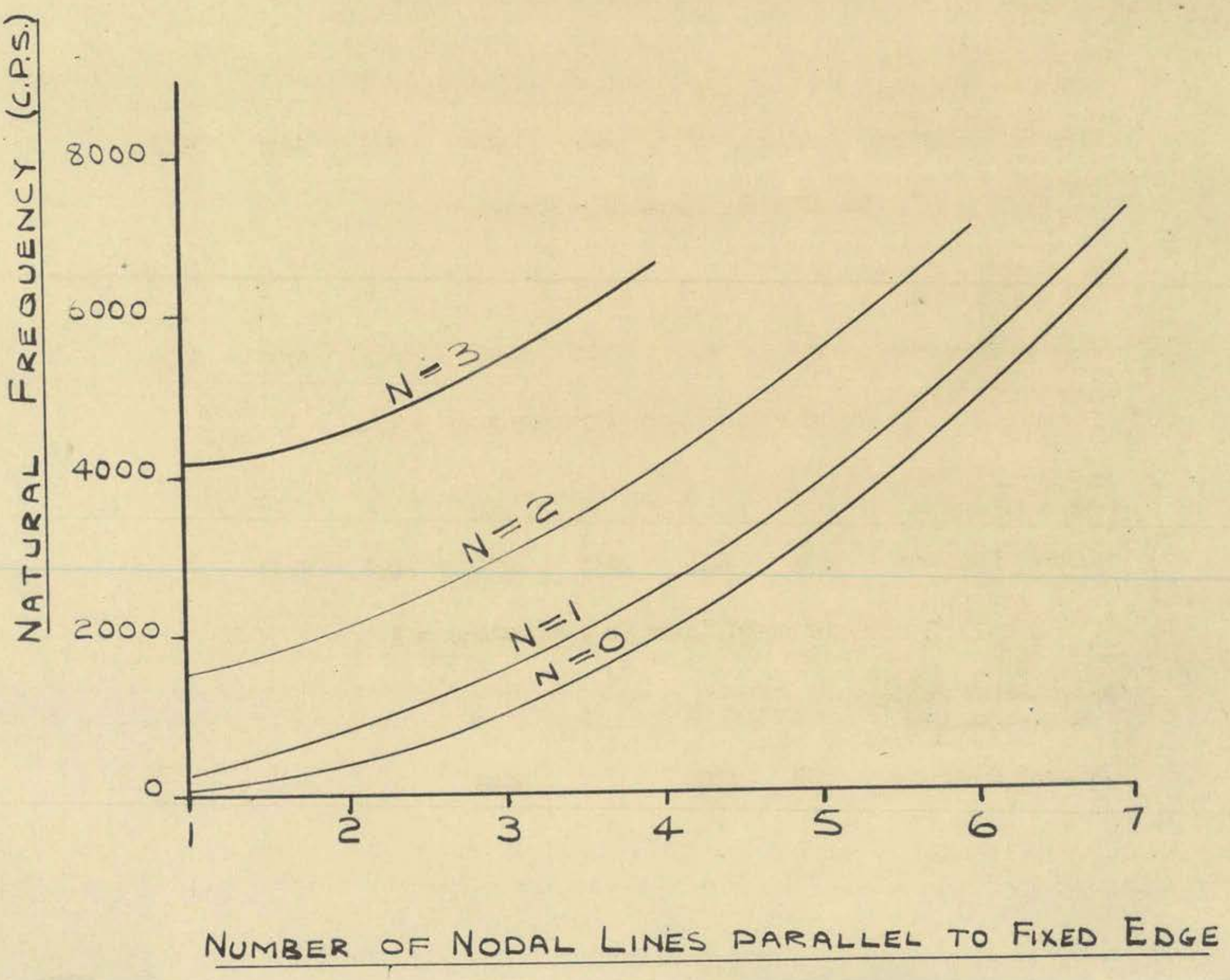
SHAPE OF NODAL PATTERNS CORRESPONDING TO NATURAL FREQUENCIES



NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

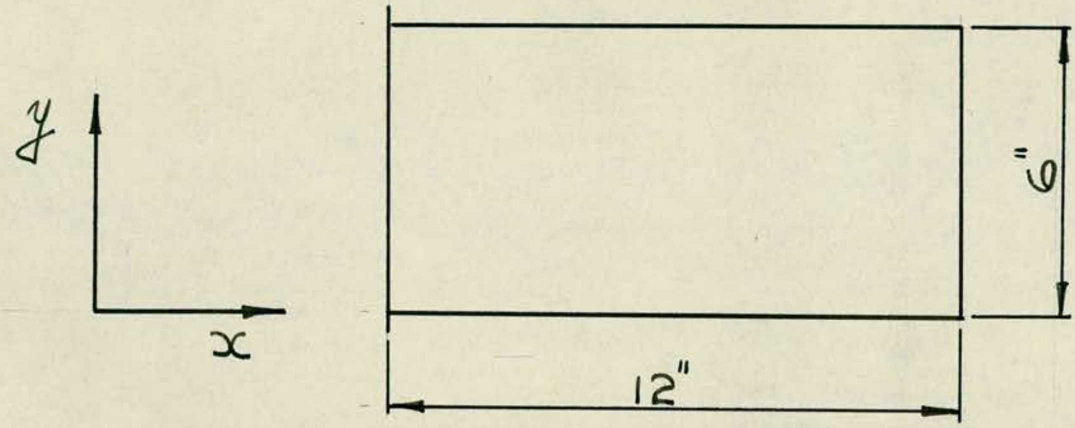
THICKNESS OF PLATE = 0·251"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE



Natural Frequencies and corresponding Modes of Vibration of Cantilevered, Right-angled Plates.

(a) Rectangular Plate



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	57.9	362	1006	1969	3251	4817	6695

No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	255	824	1558	2544	3803	5337	7235

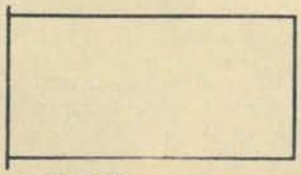
No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3	4	5	6	
Natural Frequency	1556	2111	2963	4079	5449	7053	

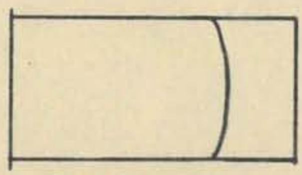
No. of nodal lines in y direction = 3

No. of nodal lines in x direction	1	2	3	4			
Natural Frequency	4141	4590		6608			

SHAPE OF NODAL PATTERNS CORRESPONDING TO NATURAL FREQUENCIES



57·9 C.P.S.



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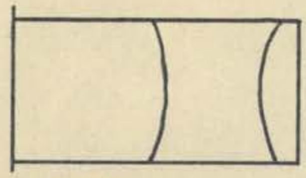

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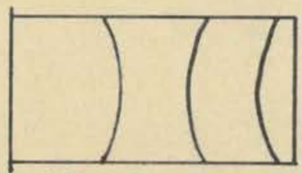

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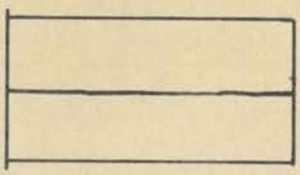

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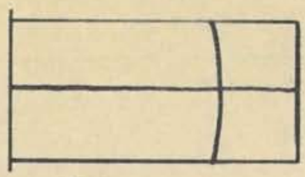

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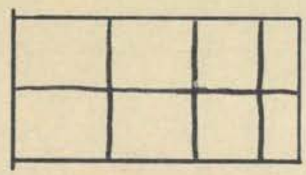

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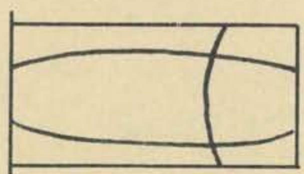

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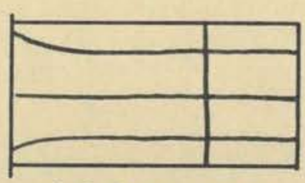
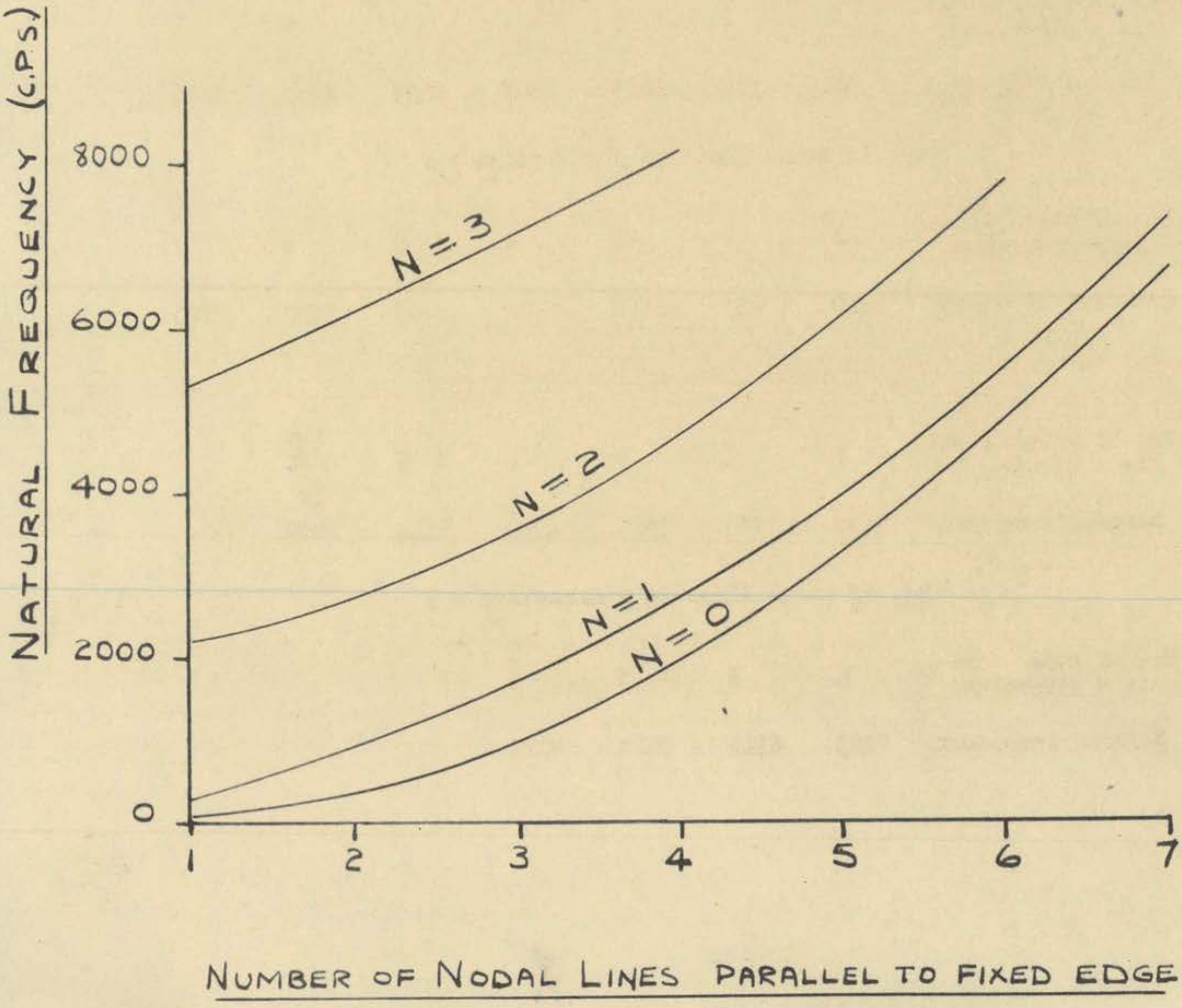


4590 C.P.S.

NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

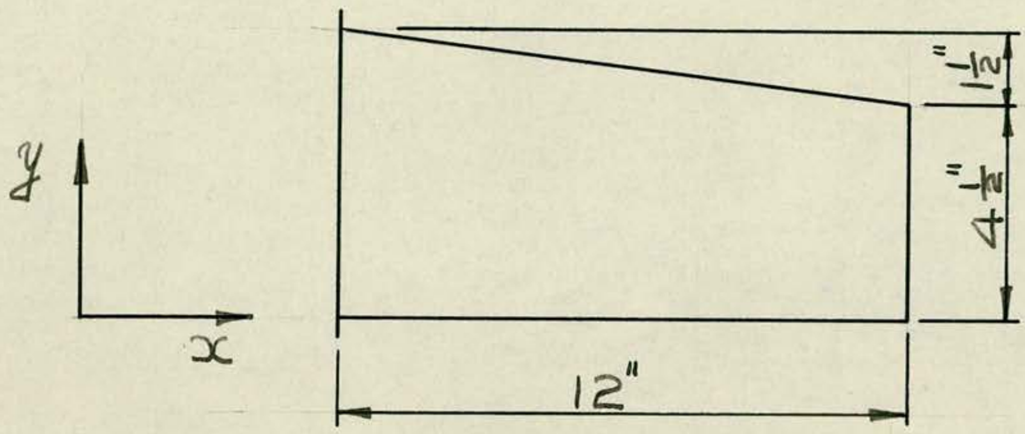
THICKNESS OF PLATE = 0·251"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE



(b) Right-angled Trapezoidal Plates

Plate No. 1



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	62.4	368	1015	1967	3239	4855	6691

No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	319	930	1714	2724	3988	5519	7325

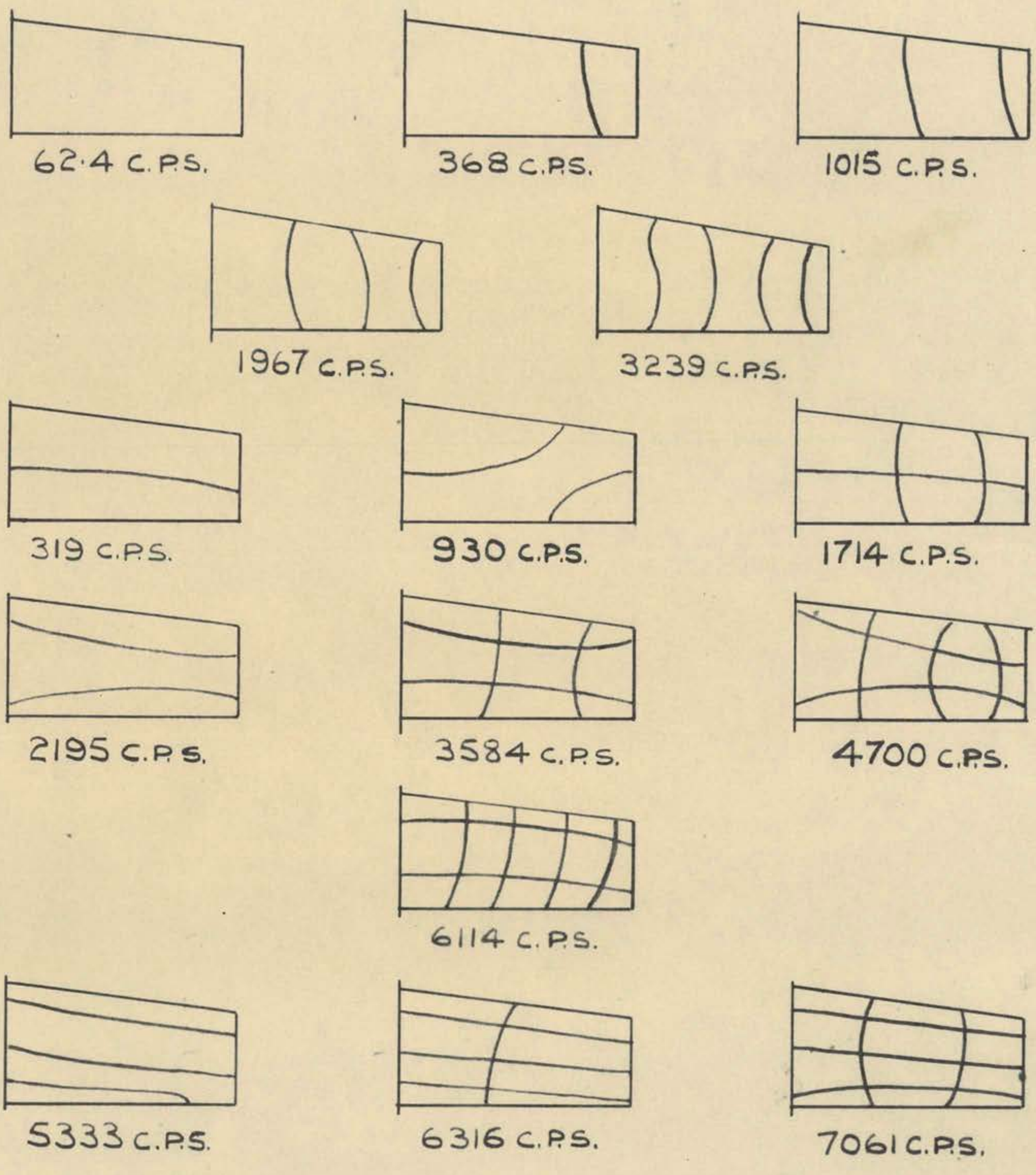
No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3	4	5	6
Natural Frequency	2195	2729	3584	4700	6114	7800

No. of nodal lines in y direction = 3

No. of nodal lines in x direction	1	2	3	4
Natural Frequency	5333	6316	7061	8115

Shape of Nodal Patterns corresponding to Natural Frequencies



NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·251"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE

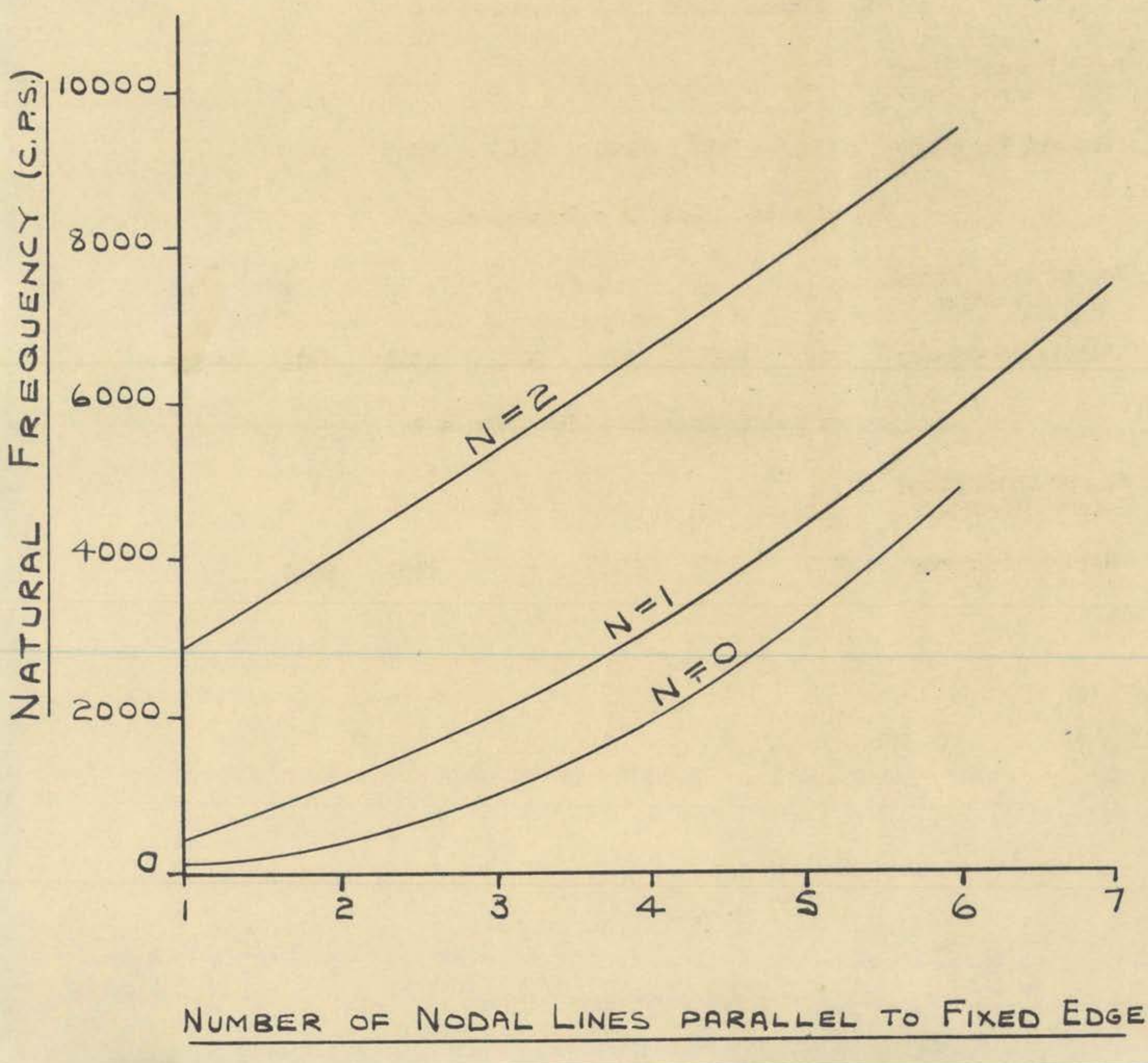
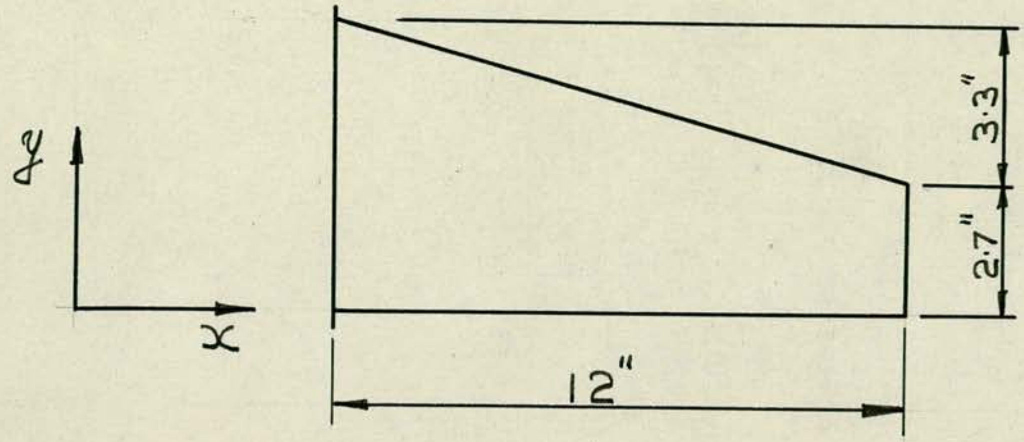


Plate No. 2



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6
Natural Frequency	71.6	376	1000	1905	3327	4857

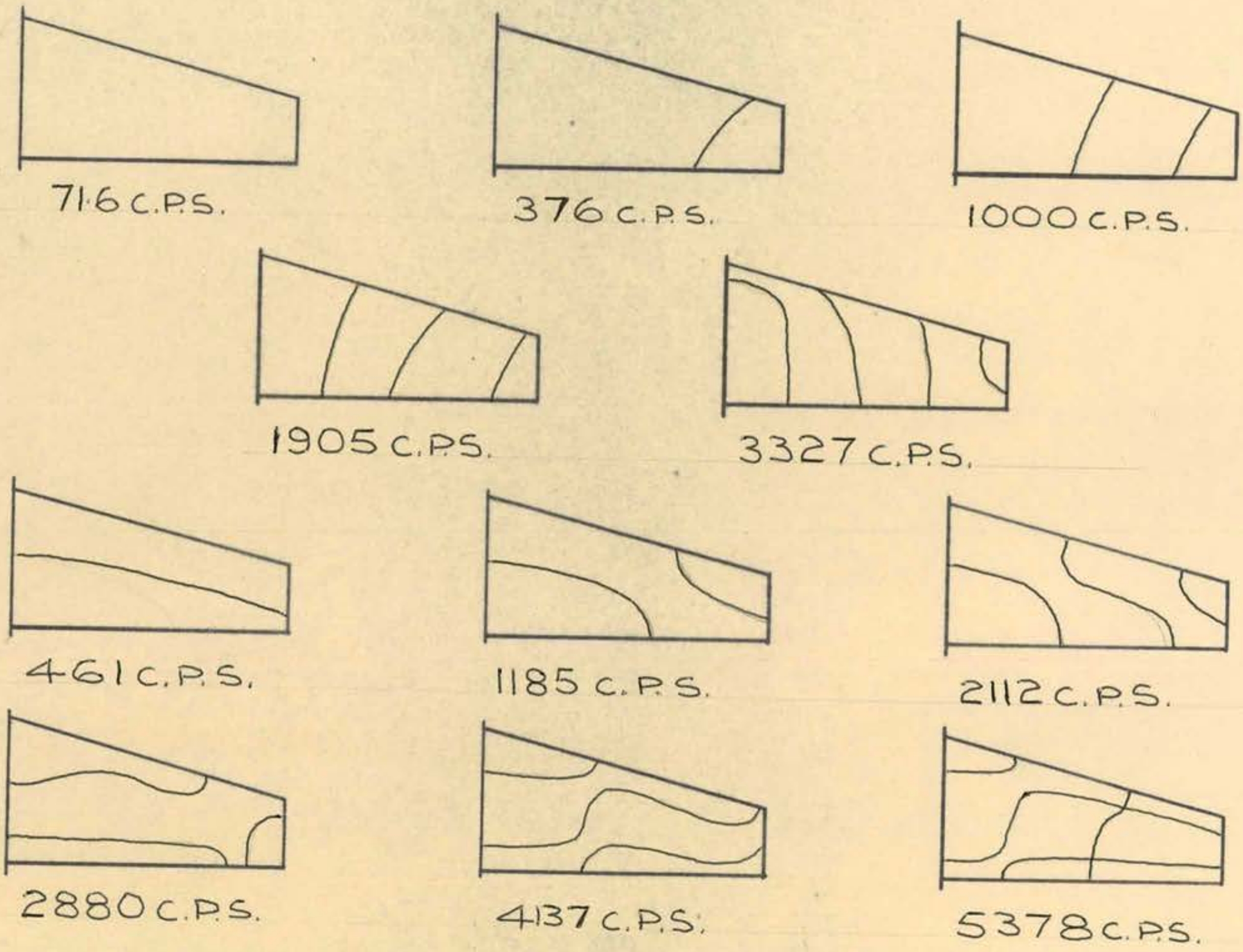
No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	461	1185	2112	3034	4381	5911	7422

No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3	4	5	6
Natural Frequmcy	2880	4137	5378		7970	9440

Shape of Nodal Patterns corresponding to Natural Frequencies



NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·251"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE

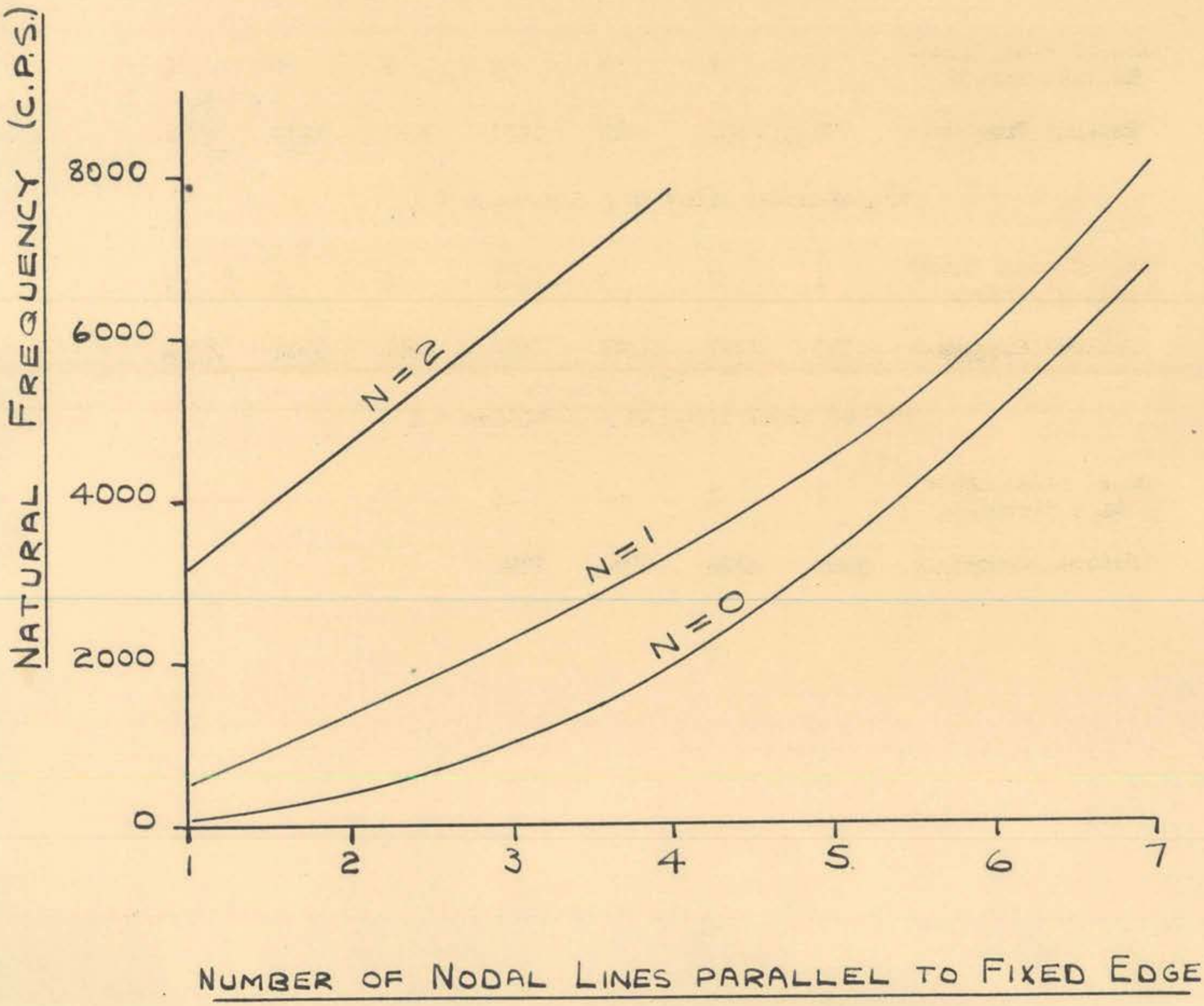
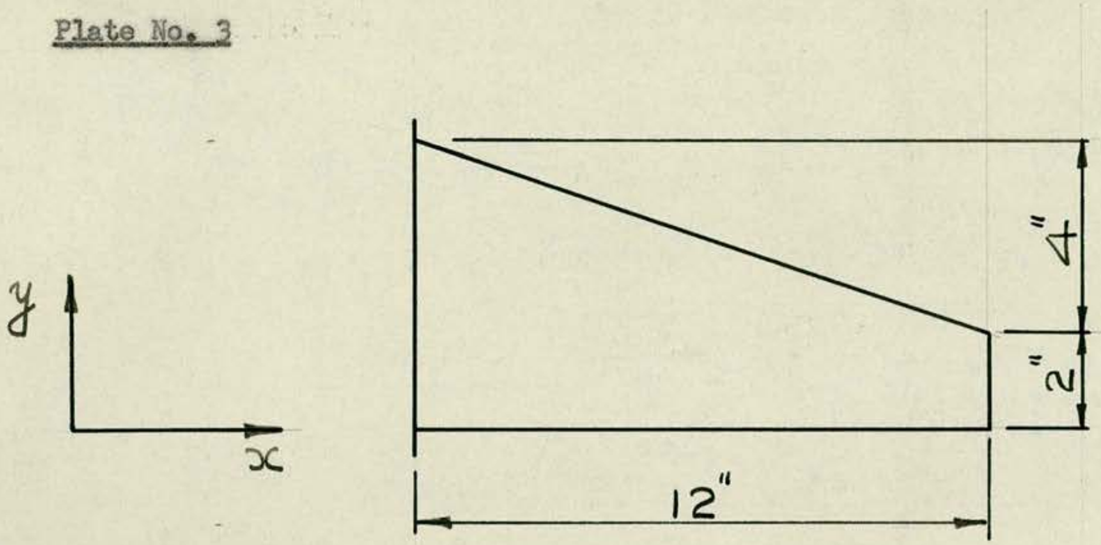


Plate No. 3



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	76.4	392	1013	1923	3067	5019	6801

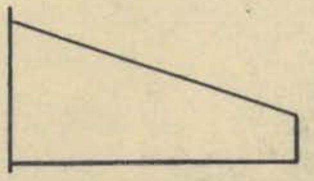
No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	535	1323	2291	3515	4461	6002	8054

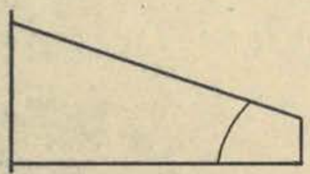
No. of nodal lines in y direction = 2

No of nodal lines in x direction	1	2	3	4
Natural Frequency	3168	4706	6306	7796

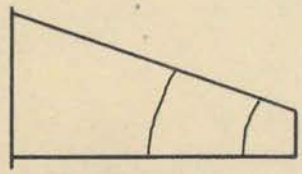
Shape of Nodal Patterns corresponding to Natural Frequencies



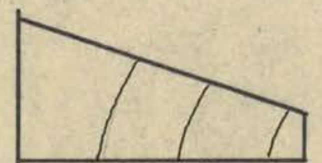
76·4 C.P.S.



392 C.P.S.



1013 C.P.S.



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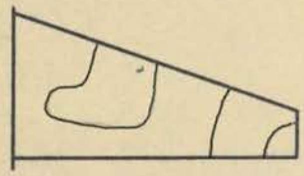
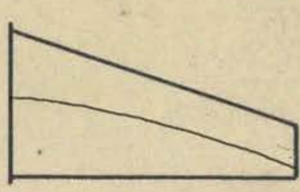
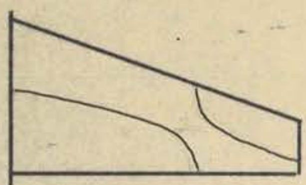


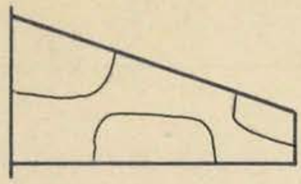
3067 C.P.S.



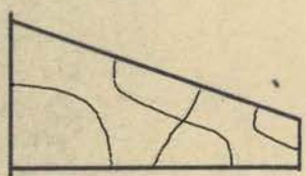
535 C.P.S.



1323 C.P.S.



2291 C.P.S.



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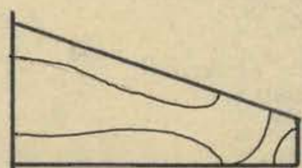

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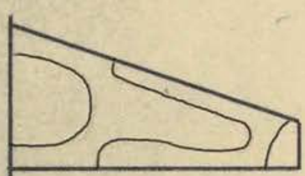
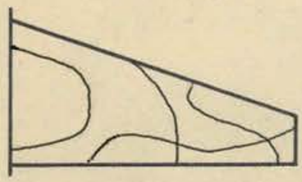


4706 C.P.S.



6306 C.P.S.

7796 C.P.S.

NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·251"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE

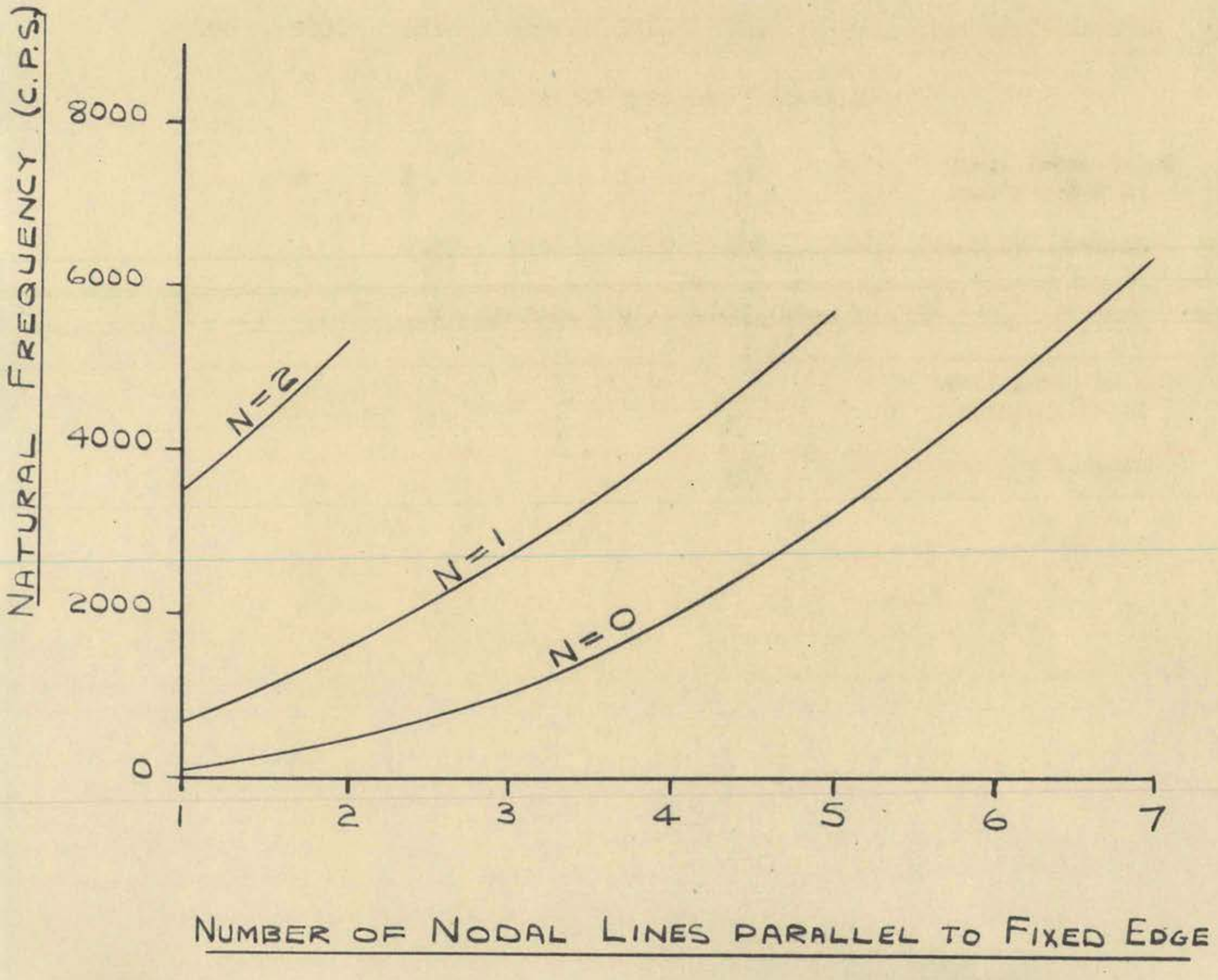
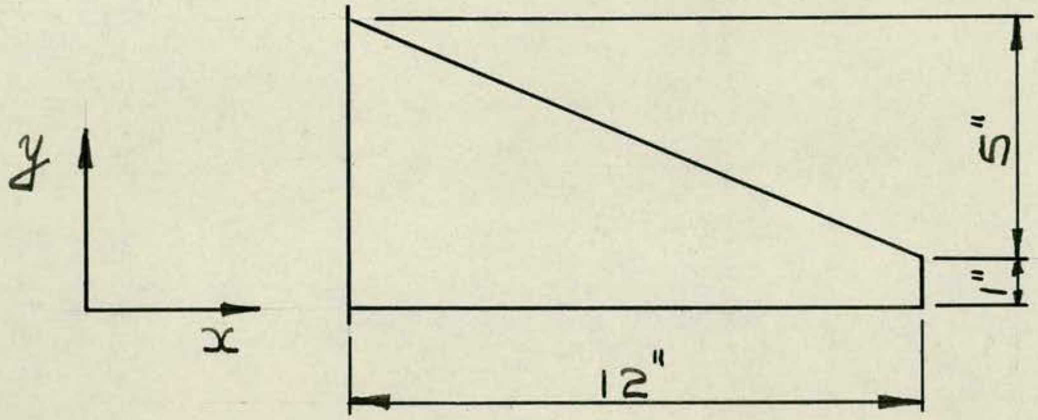


Plate No. 4



No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	87.9	409	1032	1959	3164	4660	6361

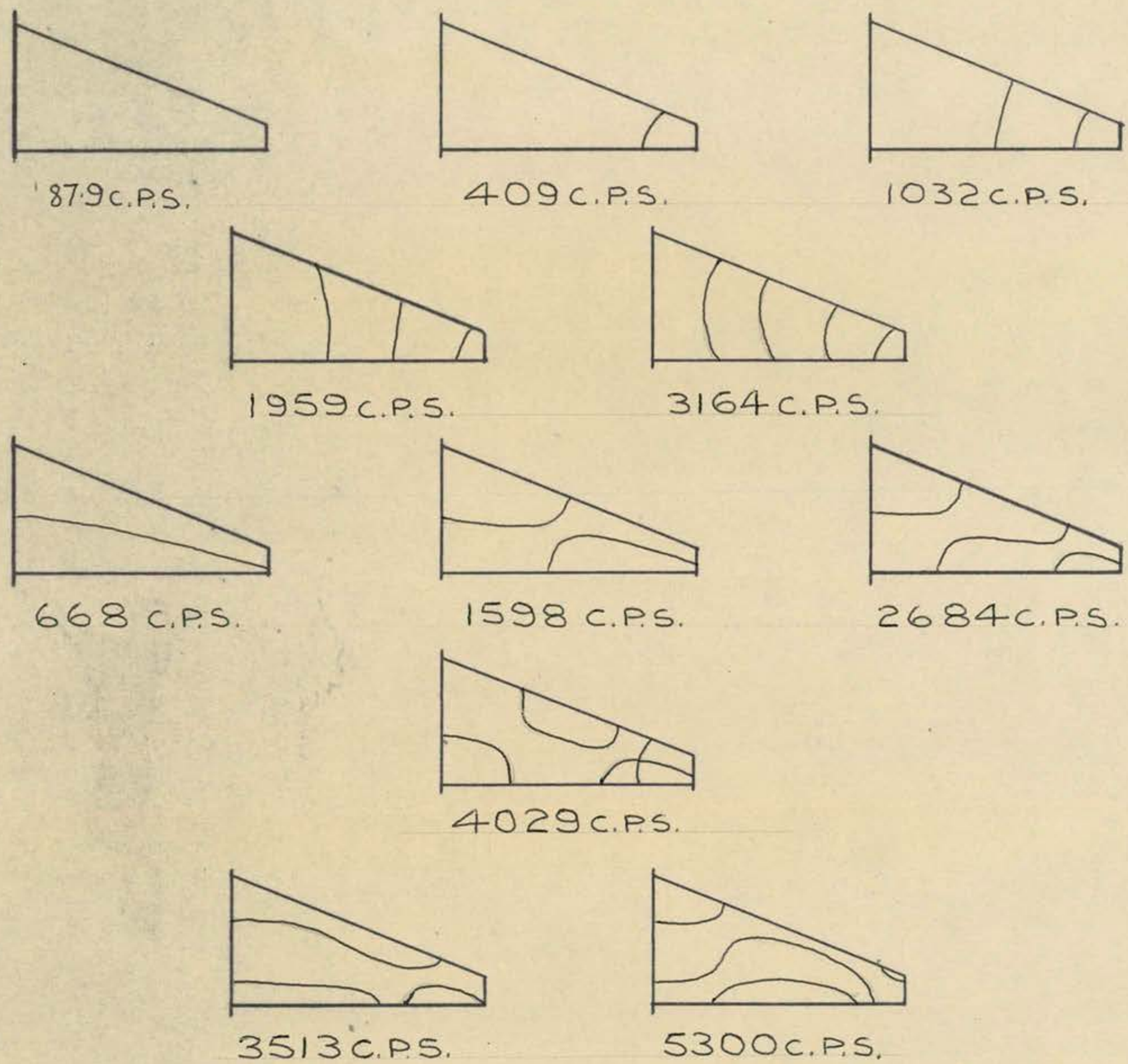
No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	668	1598	2684	4029	5689		

No. of nodal lines in y direction = 2

No. of nodal lines in x direction	1	2	3	4
Natural Frequency	3513	5300		

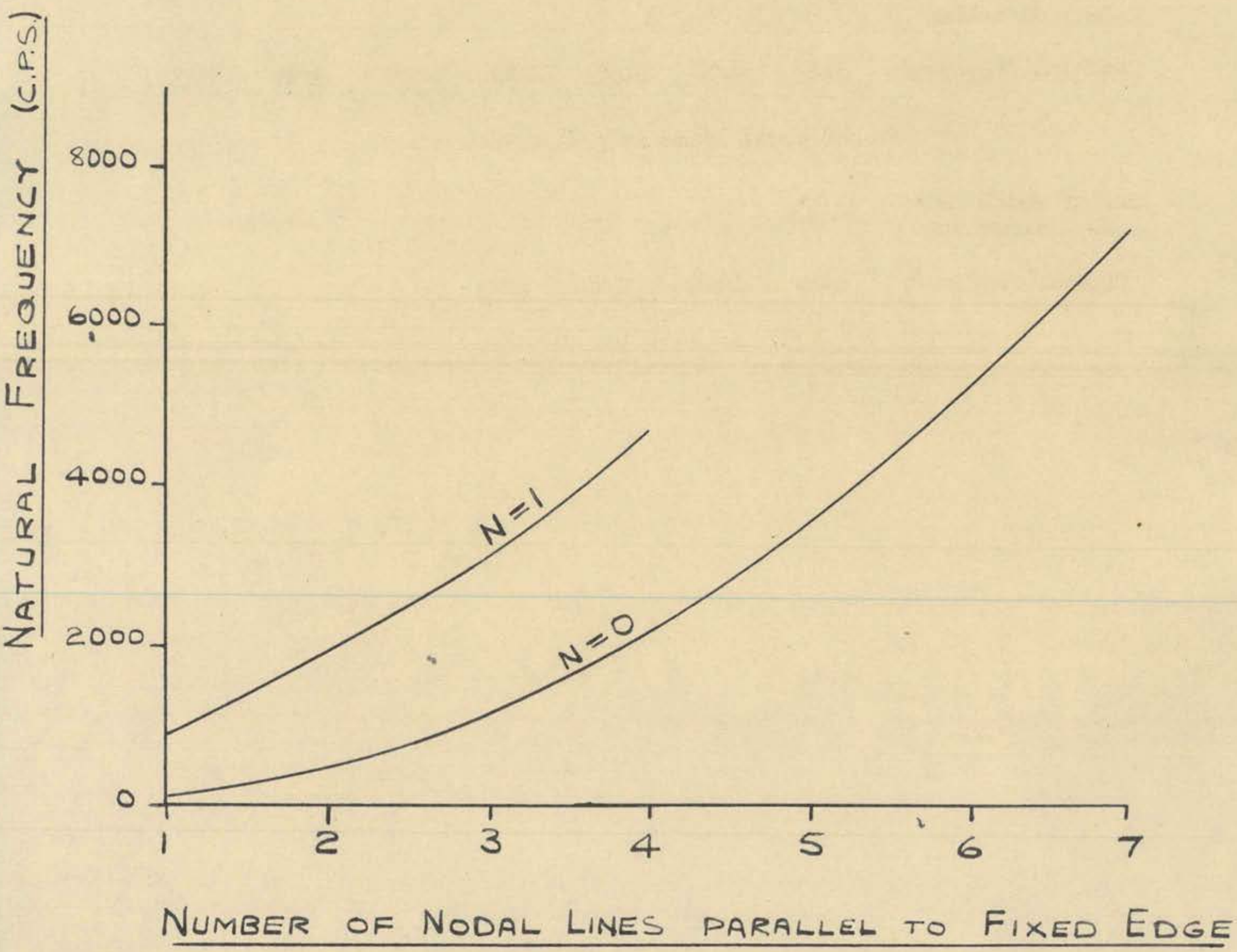
Shape of Nodal Patterns corresponding to Natural Frequencies



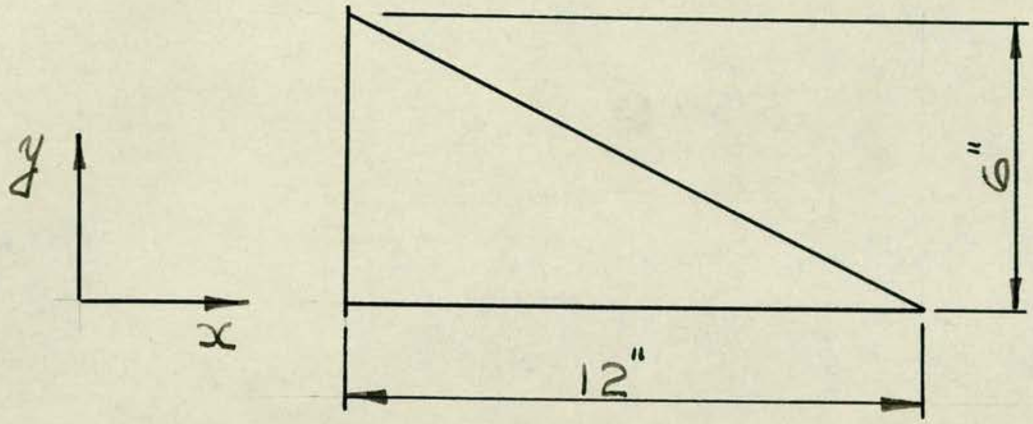
NATURAL FREQUENCY TO A BASE OF NUMBER OF NODAL LINES PARALLEL TO FIXED EDGE

THICKNESS OF PLATE = 0·251"

N = NUMBER OF NODAL LINES PERPENDICULAR TO FIXED EDGE



(c) Right-angled Triangular Plate



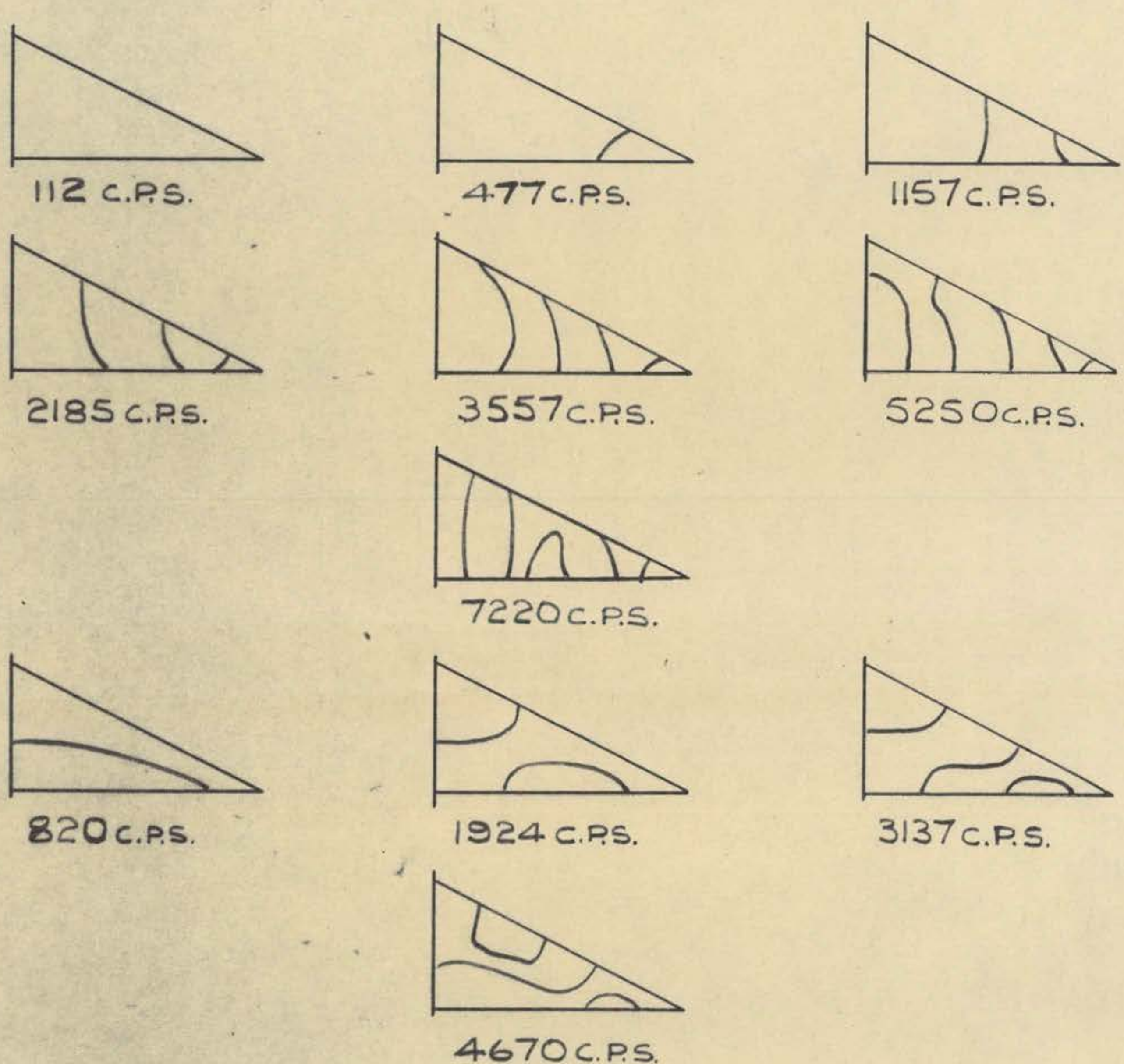
No. of nodal lines in y direction = 0

No. of nodal lines in x direction	1	2	3	4	5	6	7
Natural Frequency	112	477	1157	2185	3557	5250	7220

No. of nodal lines in y direction = 1

No. of nodal lines in x direction	1	2	3	4
Natural Frequency	820	1924	3137	4670

SHAPE OF NODAL PATTERNS CORRESPONDING TO NATURAL FREQUENCIES.



CHAPTER III

THE RAYLEIGH-RITZ METHOD OF CALCULATING NATURAL FREQUENCIES.

The exact solution of the equation of motion

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \rho\frac{h\omega^2}{gD}W \qquad 3.1$$

of a vibrating plate is known only for a rectangular plate with all edges simply supported and for a rectangular plate with two opposite edges simply supported and the other two edges having any other conditions. For a rectangular plate having any other combination of boundary conditions and for other plate shapes, it is necessary to employ an approximate method to obtain a solution to the problem. Of the approximate methods which may be used, one of the best known is the Rayleigh-Ritz method, and consists of equating the maximum potential energy and the maximum kinetic energy of the vibrating plate to obtain an expression for its natural frequencies.

The maximum potential energy of a thin, flat plate of uniform thickness which is subjected to lateral, harmonic vibrations of amplitude W and frequency ω is given by,

$$U_{max} = \frac{D}{2}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right]dy\,dx \qquad 3.2$$

and the maximum kinetic energy by

$$T_{max} = \rho\frac{h\omega^2}{2g}\int_{x_1}^{x_2}\int_{y_1}^{y_2} W^2\,dy\,dx \qquad 3.3$$

the integrations being taken over the area of the plate.

Hence

$$\omega^2 = U_{max}\Big/\rho\frac{h}{2g}\int_{x_1}^{x_2}\int_{y_1}^{y_2} W^2\,dy\,dx \qquad 3.4$$

Using the Rayleigh-Ritz method to calculate the natural frequencies, the deflected form of the vibrating plate is assumed to be

$$W = a_1 f_1(x,y) + a_2 f_2(x,y) + - - - - + a_n f_n(x,y) \qquad 3.5$$

in which each of the functions $f_m(x,y)$ must satisfy the geometric boundary conditions but need not satisfy the natural boundary conditions of the plate. Equation 3.5 is substituted in equation 3.4, and the resulting expression differentiated with respect to each of the constants a_m and equated to zero giving a series of equations of the form

$$\frac{\partial}{\partial a_m}\left[U_{max} - \frac{\rho h \omega^2}{g D}\int_{x_1}^{x_2}\int_{y_1}^{y_2} W^2 \, dy \, dx\right] = 0 \qquad 3.6$$

which become

$$\left.\begin{aligned}
&(c_{11} - \lambda k_{11})a_1 + - - - - - - - - + (c_{1n} - \lambda k_{1n})a_n = 0 \\
&(c_{21} - \lambda k_{21})a_1 + - - - - - - - + (c_{2n} - \lambda k_{2n})a_n = 0 \\
&- - - - - - - - - - - - - - \\
&(c_{n1} - \lambda k_{n1})a_1 + - - - - - + (c_{nn} - \lambda k_{nn})a_n = 0
\end{aligned}\right\} \qquad 3.7$$

in which λ is a non-dimensional frequency factor.

The natural frequencies are obtained by equating to zero the determinant of the coefficients of the constants a_m in equations 3.7.

Hence
$$\begin{vmatrix}
(c_{11} - \lambda k_{11}) & (c_{12} - \lambda k_{12}) & - - - - - & (c_{1n} - \lambda k_{1n}) \\
(c_{21} - \lambda k_{21}) & (c_{22} - \lambda k_{22}) & - - - - - - & (c_{2n} - \lambda k_{2n}) \\
- - - & - - - & - - - & - - - \\
(c_{n1} - \lambda k_{n1}) & (c_{n2} - \lambda k_{n2}) & - - - & (c_{nn} - \lambda k_{nn})
\end{vmatrix} = 0 \qquad 3.8$$

from which the values of λ and hence the natural frequencies, ω are obtained by obtaining the roots of the resulting polynomial,

3.9

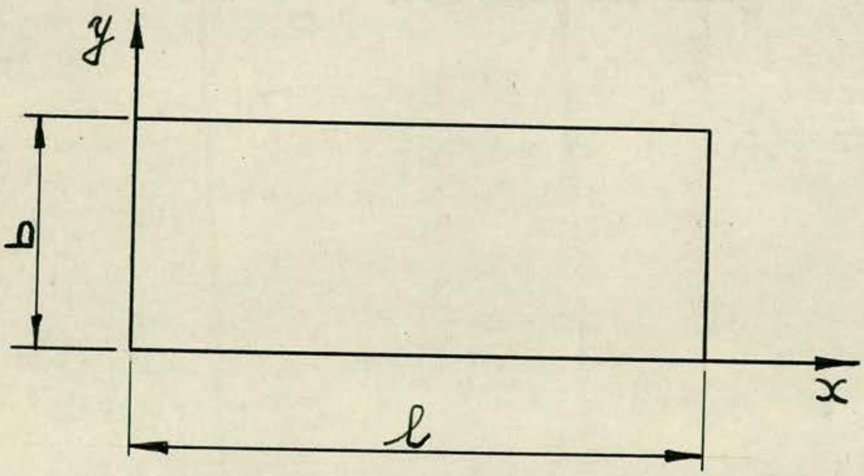
$$\alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \text{- - - -} + \alpha_1 \lambda + \alpha_0 = 0 \qquad 3.9$$

The Rayleigh-Ritz method has been used successfully by Young (1950) and Barton (1951) to calculate the natural frequencies of rectangular plates having various combinations of boundary conditions. Each of these authors used a series consisting of a combination of the characteristic functions, which represent the normal modes of vibration of uniform beams, to represent the deflected form of the vibrating plate. The deflected form of the vibrating plate, is therefore, assumed to be of the form

$$\begin{aligned} W = & A_{00}\phi_0(x)\theta_0(y) + A_{01}\phi_0(x)\theta_1(y) + \text{- - -} + A_{0n}\phi_0(x)\theta_n(y) \\ & + A_{10}\phi_1(x)\theta_0(y) + A_{11}\phi_1(x)\theta_1(y) + \text{- - -} + A_{1n}\phi_1(x)\theta_n(y) \\ & \text{— — — — — — — — — — —} \\ & + A_{n0}\phi_n(x)\theta_0(y) + A_{n1}\phi_n(x)\theta_1(y) + \text{- - -} + A_{nn}\phi_n(x)\theta_n(y) \end{aligned} \qquad 3.10$$

in which $\phi_0(x), \phi_1(x)$ - - - and $\theta_0(y), \theta_1(x)$ - - - are the appropriate beam functions depending on the boundary conditions of the plate.

Figure 3.1



If, for example, it is required to calculate the natural frequencies of a plate (Figure 3.1) fixed at x = 0 and free at x = l, y = 0 and y = b, $\phi_1(x), \phi_2(x)$ - - - would be the functions which represent the normal modes of vibration of a uniform cantilevered beam and $\theta_0(y), \theta_1(y)$ - - would be the functions which represent the normal modes of vibration of a uniform beam free at each end.

The normal modes of vibration of a uniform cantilevered beam are given by the expression

$$\phi_m(x) = \cosh \frac{\varepsilon_m x}{l} - \cos \frac{\varepsilon_m x}{l} - \alpha_m \left(\sinh \frac{\varepsilon_m x}{l} - \sin \frac{\varepsilon_m x}{l} \right) \qquad 3.11$$

where m = 1, 2, 3 . . . .

The normal modes of vibration of a uniform beam which is free at each end are given by the following expressions

$$\theta_0(y) = 1 \qquad 3.12$$

$$\theta_1(y) = \left(1 - \frac{2y}{b}\right) \qquad 3.13$$

$$\theta_m(y) = \cosh \frac{\beta_m y}{b} + \cos \frac{\beta_m y}{b} - \gamma_m \left(\sinh \frac{\beta_m y}{b} + \sin \frac{\beta_m y}{b} \right) \qquad 3.14$$

where m = 2, 3, 4 . . . .

In these expressions, 0, 1, 2, 3 . . . are the number of node points on the beam for the particular mode at which it is vibrating and ε_m, α_m, β_m and γ_m are constants depending on the mode and the end conditions. (Timoshenko, 1955, p. 338)

Warburton (1954) also used the Rayleigh-Ritz method for the calculation of the natural frequencies of rectangular plates. He used the method to derive a simple, approximate expression which can be used to calculate the natural frequencies of all modes of vibration of thin, flat rectangular plates, of uniform thickness, having any combination of fixed, simply supported or free edges. He assumed that the waveforms of plates and beams are similar, and that the nodal patterns corresponding to the natural frequencies of rectangular plates take the form of lines which are approximately parallel to the edges of the plate. The expression for the de-

deflected form, which is used to derive the frequency expression, is, therefore, a combination of the appropriate functions which represent the normal modes of vibration of uniform beams.

The nodal pattern corresponding to any natural frequency is defined by m/n, where m and n are the number of nodal lines in the x and y directions respectively. When an edge is fixed or simply supported it is considered to be a nodal line. Thus if two opposite edges are fixed or simply supported, m or n must be at least two, and if one edge is fixed or simply supported, m or n must be at least one.

If it is required to calculate the natural frequency of the mode with four nodal lines in the x-direction and two nodal lines in the y-direction of a rectangular plate (Figure 3.1) which is fixed at x = 0 and x = ℓ and free at y = 0 and y = b, the expression for the deflected form of the vibrating plate would be,

$$W = A\,\phi_4(x)\,\theta_2(y) \qquad 3.15$$

In equation 3.15 A is a constant, $\phi_4(x)$ is the deflected form of the mode with four node points, of a beam fixed at both ends and $\theta_2(y)$ is the deflected form of the mode with two node points, of a beam free at both ends. Equation 3.15 is substituted in the expressions for the maximum kinetic energy and the maximum potential energy of the vibrating plate, and the natural frequency of the mode being considered is obtained by equating these two expressions.

For any combination of boundary conditions, therefore, the natural frequency, corresponding to any mode of vibration, can be obtained, and in each case a non-dimensional frequency factor λ_1, proportional to the natural frequency can be derived,

$$\lambda_1^2 = \frac{\rho h \omega^2 \ell^4}{\pi^4 g D} \qquad 3.16$$

For all combinations of fixed, simply supported and free boundary conditions

$$\lambda_1^2 = G_x^4 + G_y^4 \frac{\ell^4}{b^4} + 2\frac{\ell^2}{b^2}\left[\mu H_x H_y + (1-\mu) J_x J_y\right] \qquad 3.17$$

in which the coefficients $G_x : G_y : H_x : H_y : J_x : J_y$ depend on the nodal patterns and the boundary conditions.

For some of the combinations of boundary conditions Warburton found no reference to known frequencies. For the remainder, the method has proved to be highly successful for almost every combination of boundary conditions. The values of the natural frequencies obtained using the simple approximate method compare favourably with values obtained previously by more exact analysis or by experiment.

The main exception occurs when the method is used to calculate the natural frequencies, with one nodal line perpendicular to the fixed edge, of a rectangular plate with one edge fixed and the others free. Suppose for example that it is required to calculate the natural frequencies corresponding to the modes m/1 of a rectangular plate fixed at x = 0 and free at $x = \ell$, y = 0 and y = b. The deflected form of the vibrating plate for these modes is assumed to be

$$W = A\,\phi_m(x)\,\theta_1(y) \qquad 3.18$$

The values obtained for the natural frequencies, corresponding to these modes, are much higher than the values obtained by more exact analysis or by experiment. The biggest discrepancy occurs for the mode 1/1. The value of the natural frequency obtained for that mode can be greatly improved, however, if the deflected form of the vibrating plate is assumed to be,

$$W = A\,\phi_1(x)\,\theta_1(y) + B\,\phi_2(x)\,\theta_1(y) \qquad 3.19$$

CHAPTER IV

CALCULATION OF THE NATURAL FREQUENCIES OF CANTILEVERED PLATES USING BEAM FUNCTIONS TO REPRESENT THE DEFLECTED FORM OF THE VIBRATING PLATE.

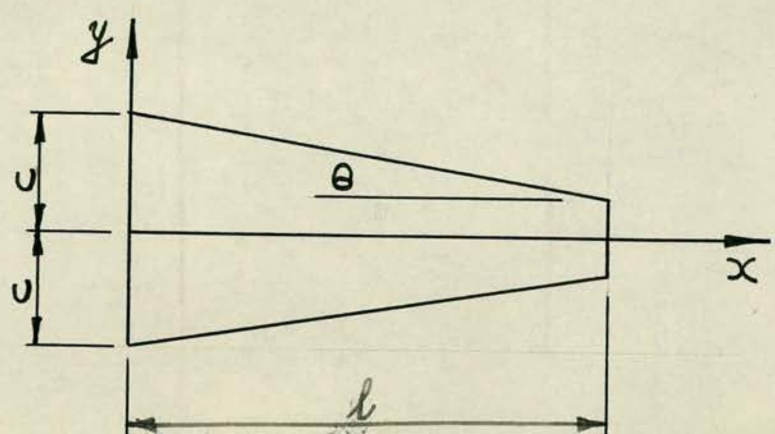
4.1 Single term approximation of the deflected form of the vibrating plate.

In view of the success of the method used by Warburton for the calculation of the natural frequencies of rectangular plates, it was decided to investigate the possibility of extending it to include thin, flat trapezoidal and triangular shaped plates of uniform thickness. It was used, therefore, in an attempt to calculate the natural frequencies of the thin, flat, cantilevered, rectangular, trapezoidal and triangular shaped plates, of uniform thickness, which were investigated experimentally. In the work that is to follow, therefore, the functions which will be used to represent the deflected form of the vibrating plate will be those which represent the normal modes of vibration of a uniform cantilevered beam and a uniform beam which is free at each end. These functions are given by equations 3.11 to 3.14.

During the calculations the symmetrical and the right-angled plates were considered separately, and the natural frequencies under consideration were divided into families, according to the shape of their nodal patterns. The natural frequencies in any particular family have the same number of nodal lines perpendicular to the fixed edge of the plate.

4.1.1 Natural Frequencies of Symmetrical Plates.

Figure 4.1



The maximum potential energy of a thin, flat plate, of uniform thickness, which is subjected to lateral, harmonic vibrations of amplitude W(x,y) and frequency ω and having a plan form which is either a rectangle, a symmetrical trapezium or an isosceles triangle, is given by the expression

$$U_{max} = \frac{D}{2}\int_0^l\int_{y_1}^{y_2}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2+2\mu\,\frac{\partial^2 W}{\partial x^2}\,\frac{\partial^2 W}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right]dy\,dx \qquad 4.1$$

and the maximum kinetic energy by the expression

$$T_{max} = \rho\frac{h\omega^2}{2g}\int_0^l\int_{y_1}^{y_2} W^2\,dy\,dx \qquad 4.2$$

where $y_1 = (x\tan\theta - c)$

and $y_2 = (c - x\tan\theta)$

By equating the expressions for the maximum potential energy and the maximum kinetic energy an expression for the natural frequencies of the plate can be obtained.

Hence $$\omega^2 = U_{max}\Big/\rho\frac{h}{2g}\int_0^l\int_{y_1}^{y_2} W^2\,dy\,dx \qquad 4.3$$

(a) <u>Calculation of the natural frequencies of the family with no nodal lines perpendicular to the fixed edge</u>.

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with no nodal lines perpendicular to the fixed edge of the cantilevered, symmetrical plates, investigated experimentally, the deflected form of the vibrating plate is assumed to be

$$W = A\,\phi_m(x)\,\theta_0(y) \qquad 4.4$$

Substituting equation 4.4 in equation 4.3, the natural frequencies of the first seven modes of the family, of the plates under consideration, were calculated. The calculated and experimental frequencies are shown in Table 1.

The percentage difference between the calculated and experimental frequencies included in Table 1, and subsequent tables, is defined as,

$$\text{Percentage Difference} = \frac{\text{Calculated Frequency} - \text{Experimental Frequency}}{\text{Experimental Frequency}} \times 100$$

The angle θ , which is used to express the variation in area of the symmetrical plates, is shown in Figure 4.1.

Variation of Natural Frequency of Modes 1/0 and 2/0 with Area of Plate

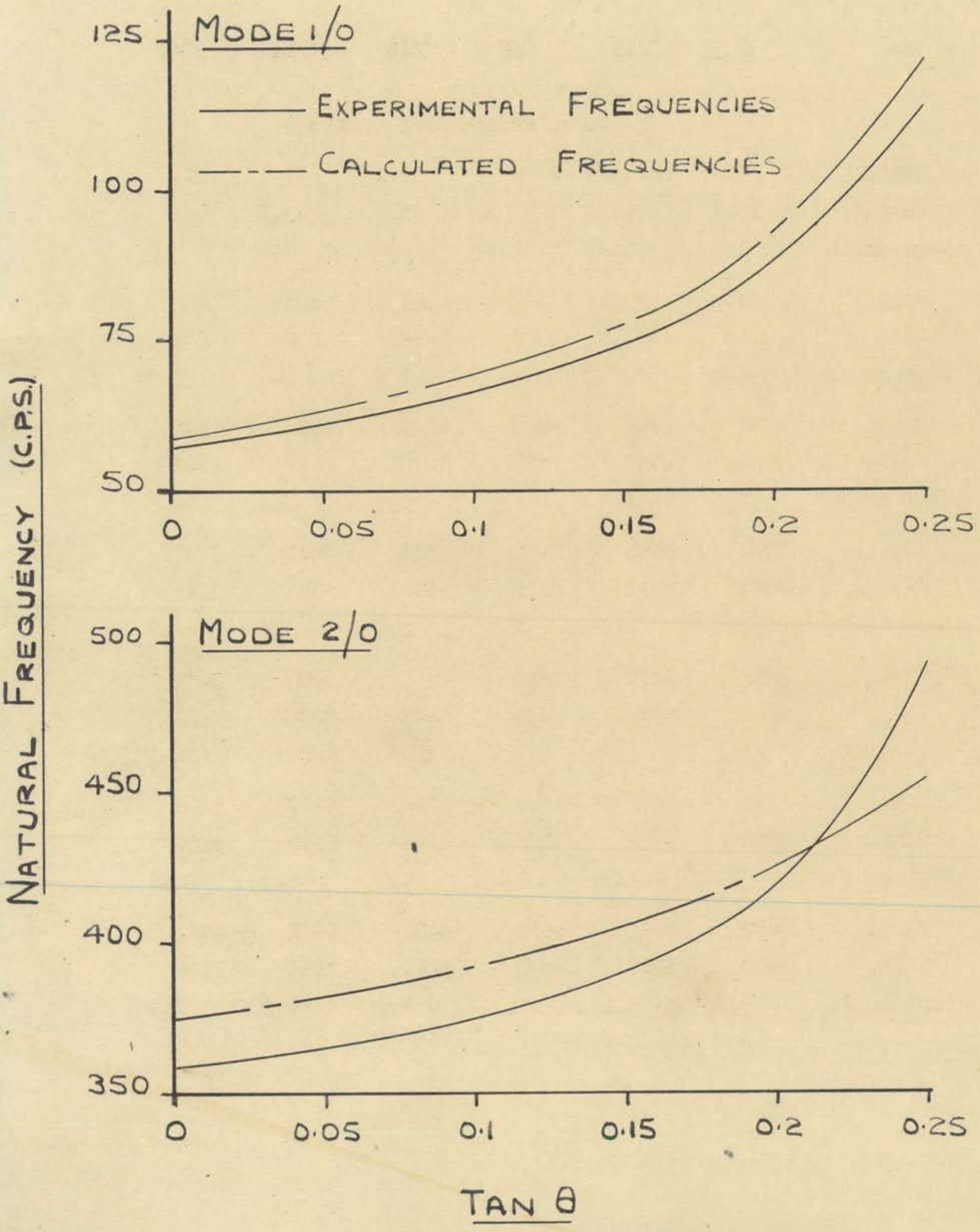


Table 1: Experimental and Calculated Natural frequencies of the family m/o of Symmetrical plates.

Assumed Deflected form for calculated frequencies:-

$$W = A\,\phi_m(x)\,\theta_0(y)$$

	tan θ	0	$^1/_{16}$	$^1/_8$	$^3/_{16}$	$^5/_{24}$	$^1/_4$
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/0	Expt.	58	63.4	70.2	83.4	90.1	114
	Calc.	59.7	65.2	73.3	87.5	95.3	122
	%age diff.	2.9	2.8	4.4	4.9	5.8	7.0
2/0	Expt.	359	369	380	409	422	494
	Calc.	374	385	398	418	428	453
	%age diff.	4.2	4.3	4.7	2.2	1.4	-8.3
3/0	Expt.	1003	1011	1022	1046	1062	1193
	Calc.	1046	1056	1069	1087	1096	1115
	%age diff.	4.3	4.4	4.6	3.9	3.5	-6.5
4/0	Expt.	1970	1966	1976	1996	2011	2202
	Calc.	2050	2059	2072	2090	2098	2118
	%age diff.	4.1	4.7	4.9	4.7	4.3	-3.8
5/0	Expt.	3255	3237	3233		3250	3485
	Calc.	3387	3398	3411	3425	3438	3457
	%age diff.	4.1	5.0	5.5		5.8	-0.8
6/0	Expt.	4823	4846	4757	4792	4807	5100
	Calc.	5062	5072	5084	5102	5110	5130
	%age diff.	5.0	4.7	6.8	6.5	6.3	0.6
7/0	Expt.	6690	6683	6668	6604	6632	6964
	Calc.	7070	7080	7092	7111	7118	7138
	%age diff.	5.7	5.9	6.4	7.7	7.3	2.5

(b) Calculation of the Natural Frequencies of the family with one nodal line perpendicular to the fixed edge.

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with one nodal line perpendicular to the fixed edge of the cantilevered, symmetrical plates, investigated experimentally, the deflected form of the vibrating plate is assumed to be,

$$W = A\,\phi_m(x)\,\theta_1(y) \qquad 4.5$$

For the calculation of the natural frequencies of the family m/1 the function $\theta_1(y)$ as expressed by equation 3.13 must be modified, due to the x-axis coinciding with the centre line of the plate. For the calculation of the natural frequencies of the family m/1, $\theta_1(y)$ will therefore be as expressed by equation 4.6.

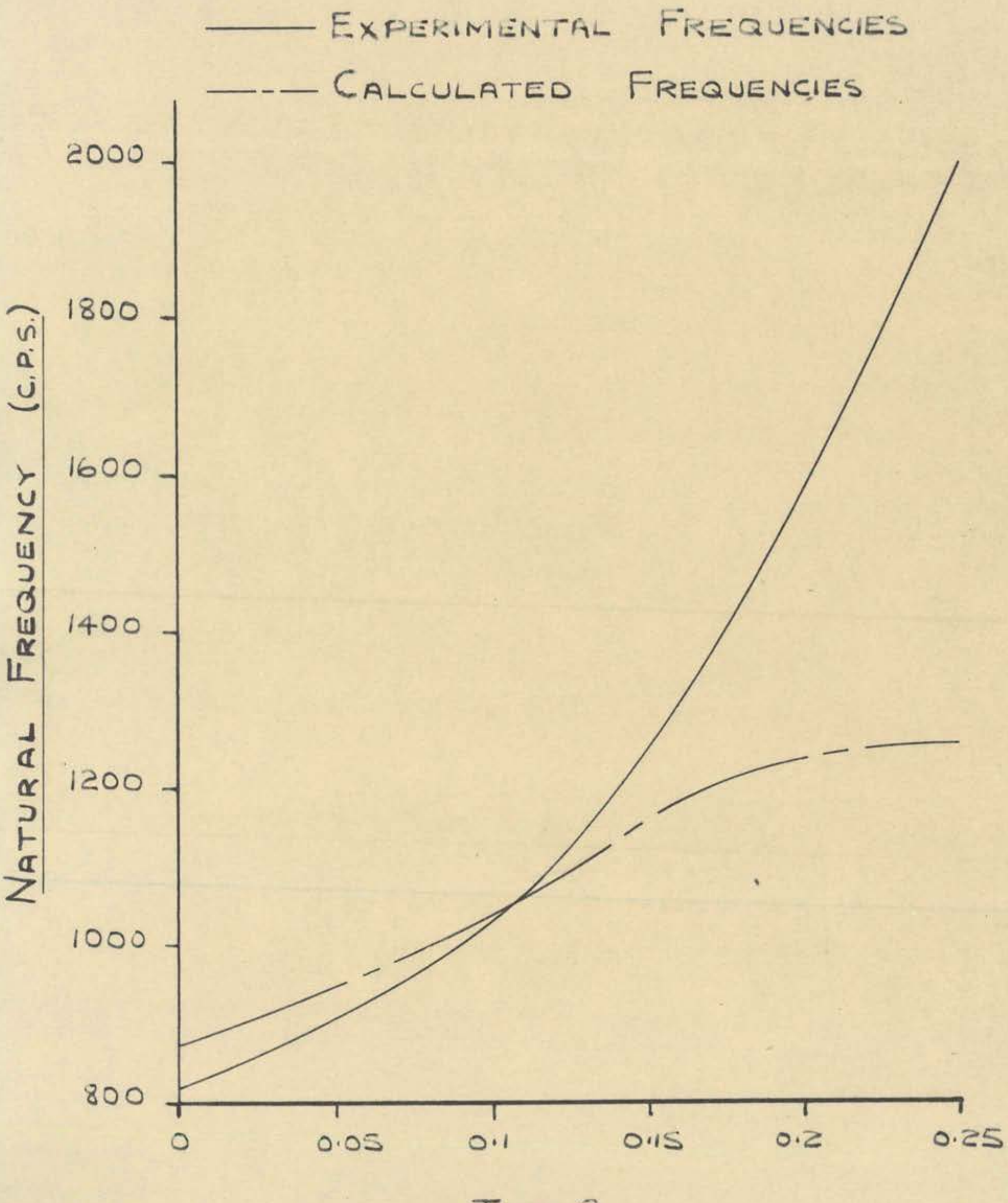
$$\text{i.e.} \quad \theta_1(y) = \frac{2y}{b} \qquad 4.6$$

Substituting equation 4.5 in equation 4.3, the natural frequencies of the first four modes of the family, of the plates under consideration, were calculated. The calculated and experimental frequencies are shown in Table 2.

Table 2: . . .

VARIATION OF NATURAL FREQUENCY OF MODE 2/1

WITH AREA OF PLATE



Variation of Natural Freqency of Mode 1/1 with Area of Plate

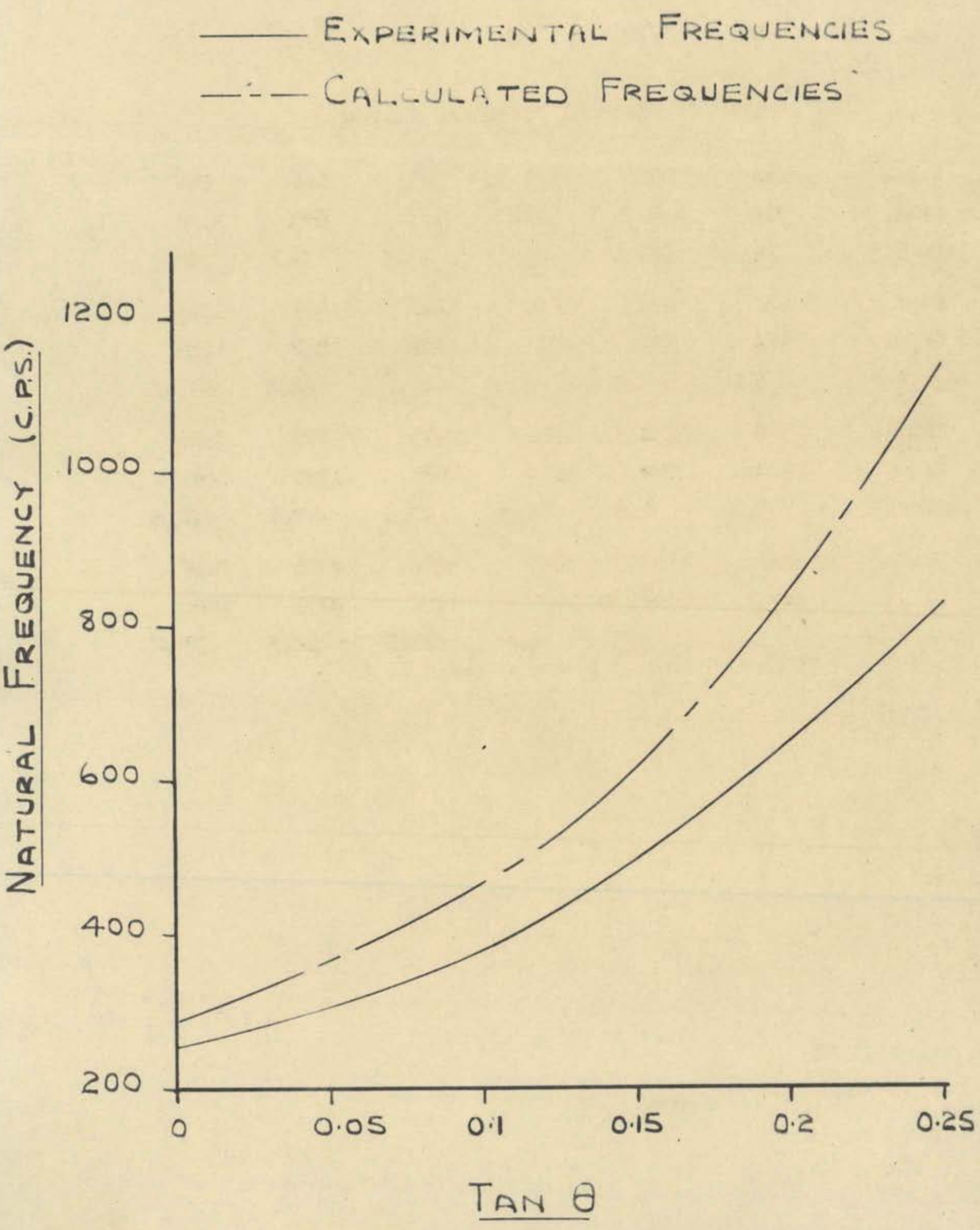


Table 2: Experimental and Calculated Natural Frequencies of the family m/1 of symmetrical plates.

Assumed deflected form for calculated frequencies:-

$$W = A\,\phi_m(x)\,\theta_1(y)$$

	$\tan\theta$	0	$^1/_{16}$	$^1/_8$	$^3/_{16}$	$^5/_{24}$	$^1/_4$
Nodal Pattern	Natural Frequency (c.p.s.)						
1/1	Expt.	256	322	430	597	666	821
	Calc.	305	390	529	773	873	1138
	%age diff.	19.1	21.1	23	29.5	31.1	38.6
2/1	Expt.	820	939	1119	1446	1606	1994
	Calc.	874	989	1083	1228	1240	1253
	%age diff.	6.6	5.3	-3.2	-15.1	-22.8	-37.2
3/1	Expt.	1558	1728	1994	2473	2722	3416
	Calc.	1608	1739	1876	1967	1980	1967
	%age diff.	3.2	0.6	-5.9	-20.5	-27.2	-42.4
4/1	Expt.	2545	2739	3079	3712	4056	5096
	Calc.	2638	2786	2938	3051	3070	3067
	%age diff.	3.6	1.7	-4.6	-17.8	-24.3	-39.8

The natural frequencies of the first two modes of the family with one nodal line perpendicular to the fixed edge of the cantilevered, symmetrical plates investigated experimentally were calculated using the Rayleigh-Ritz method and assuming the deflected form of the vibrating plate to be,

$$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y) \qquad 4.7$$

The natural frequency of the second mode of the family of these plates was calculated using the Rayleigh-Ritz method and assuming the deflected form of the vibrating plate to be,

$$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y) \qquad 4.8$$

The calculated and experimental frequencies are shown in Tables 3 and 4.

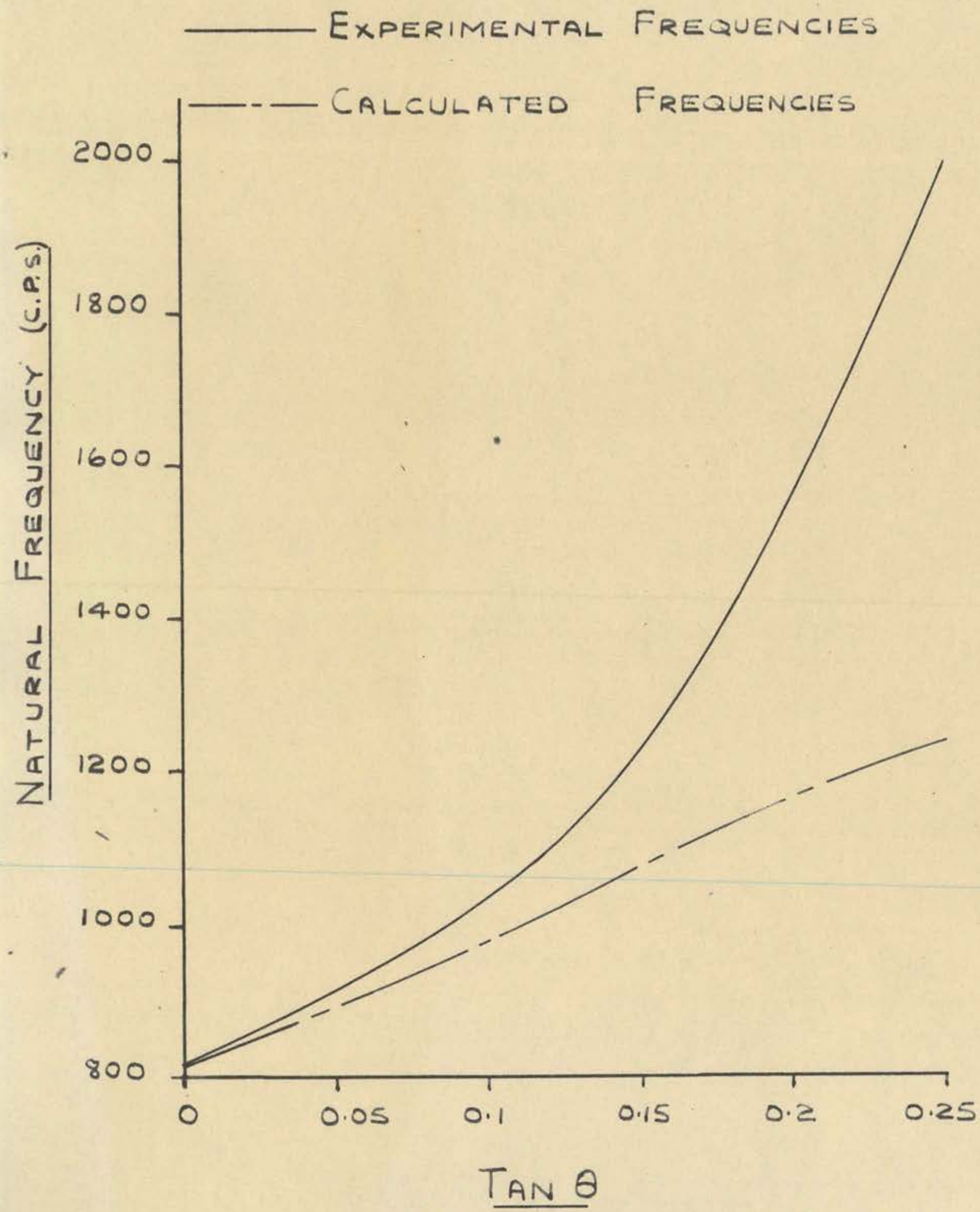
<u>Tables 3 and 4</u>: . . .

ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:-

$$W = \left[A\phi_2(x) + B\phi_3(x)\right]\theta_1(y)$$

VARIATION IN NATURAL FREQUENCY OF MODE 2/1

WITH AREA OF PLATE



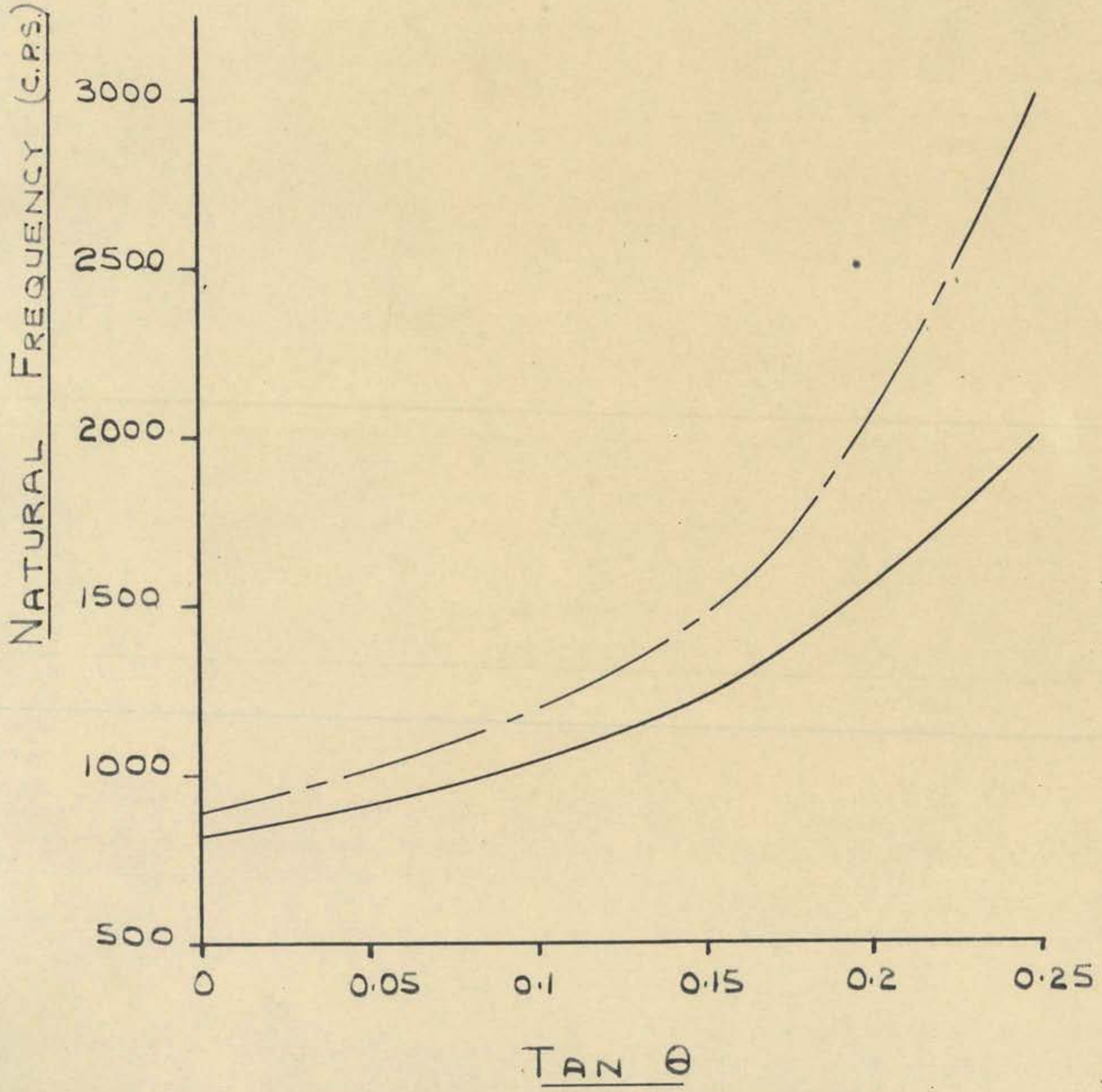
ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:-

$$W = \left[A\phi_1(x) + B\phi_2(x)\right]\theta_1(y)$$

VARIATION OF NATURAL FREQUENCY OF MODE 2/1 WITH AREA OF PLATE

——— EXPERIMENTAL FREQUENCIES

—·—— CALCULATED FREQUENCIES



ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES :-

$$W = \left[A\phi_1(x) + B\phi_2(x)\right]\theta_1(y)$$

VARIATION IN NATURAL FREQUENCY OF MODE 1/1 WITH

AREA OF PLATE

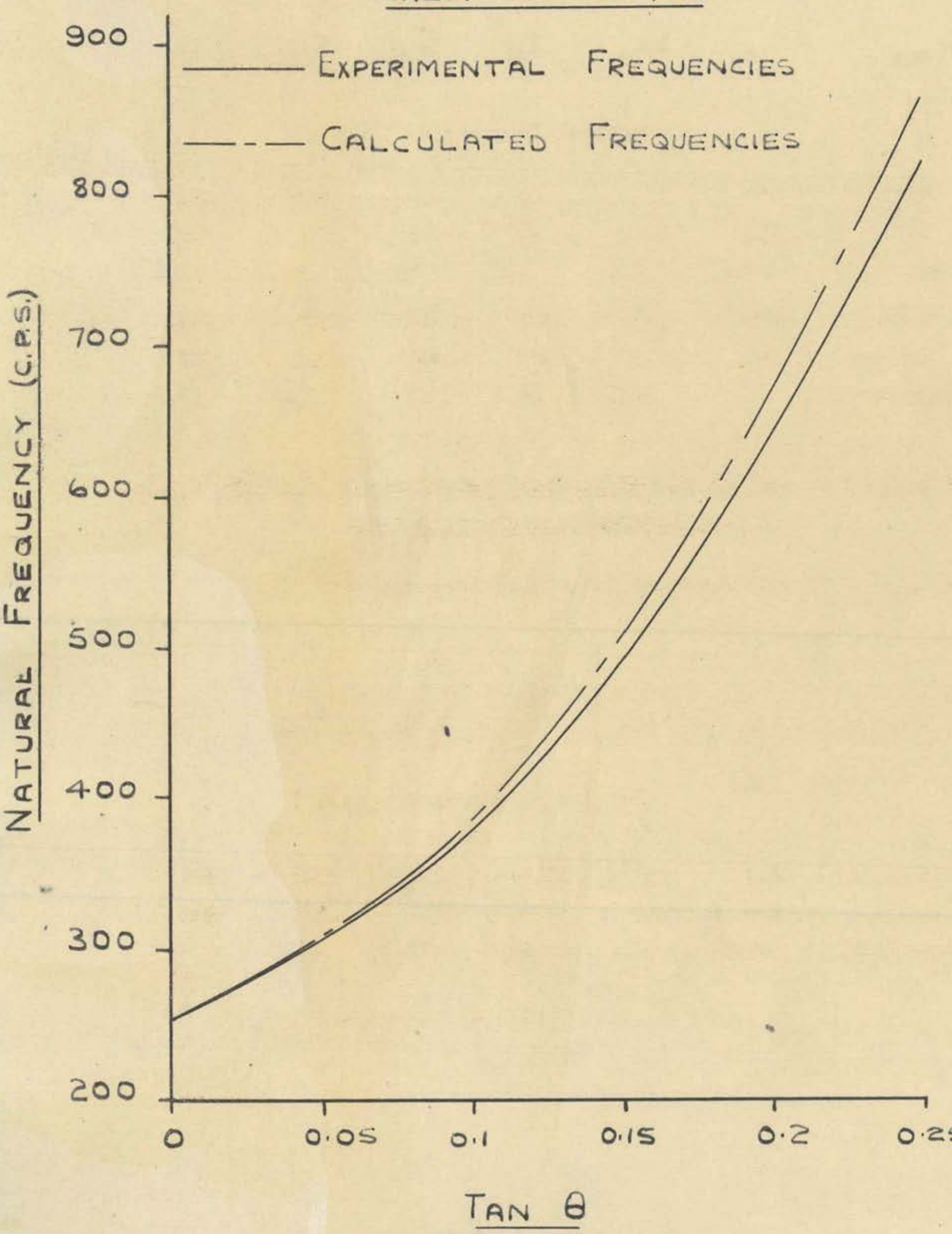


Table 3: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered Symmetrical Plates.

Assumed Deflected Form for Calculated Frequencies:-

$$W = \left[A\varphi_1(x) + B\varphi_2(x)\right]\theta_1(y)$$

	tan θ	0	1/16	1/8	3/16	5/24	1/4
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/1	Expt.	256	322	430	597	666	821
	Calc.	254	324	436	615	692	865
	%age diff.	-0.8	0.6	1.4	3.0	3.9	5.4
2/1	Expt.	820	939	1119	1446	1606	1994
	Calc.	891	1049	1307	1851	2149	3003
	%age diff.	8.7	11.7	16.8	28	33.8	50.6

Table 4: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered Symmetrical Plates.

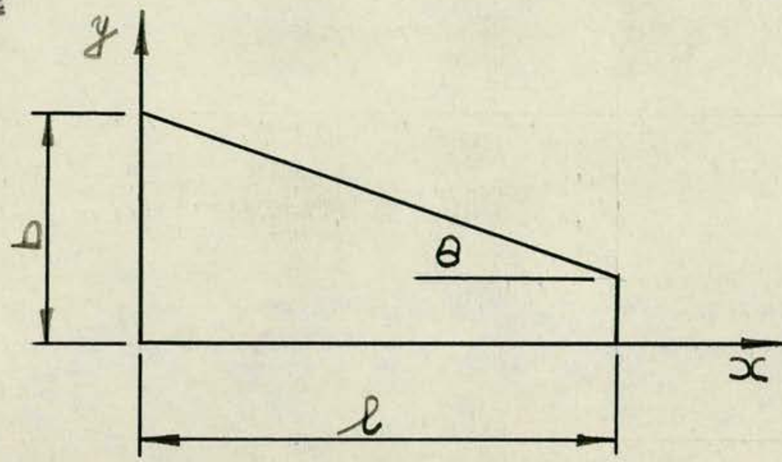
Assumed Deflected Form for Calculated Frequencies:-

$$W = \left[A\varphi_2(x) + B\varphi_3(x)\right]\theta_1(y)$$

	tan θ	0	1/16	1/8	3/16	5/24	1/4
Nodal Pattern	Natural Frequencies (c.p.s.)						
2/1	Expt.	820	939	1119	1446	1606	1994
	Calc.	816	916	1026	1141	1177	1230
	%age diff.	-0.5	-2.4	-8.3	-21.1	-26.7	-38.3

4.1.2 Natural Frequencies of Right-angled Plates.

Figure 4.2



The maximum potential energy of a thin, flat plate of uniform thickness, which is subjected to lateral, harmonic vibrations of amplitude $W(x,y)$ and frequency ω, and having a plan form which is either a rectangle, a right-angled trapezium or a right-angled triangle, is given by the expression,

$$U_{max} = \frac{D}{2}\int_0^l\int_{y_1}^{y_2}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right] dy\,dx \qquad 4.9$$

and the maximum kinetic energy by the expression

$$T_{max} = \frac{\rho h \omega^2}{2g}\int_0^l\int_{y_1}^{y_2} W^2\,dy\,dx \qquad 4.10$$

where $y_1 = 0$ and $y_2 = (b - x\tan\theta)$.

By equating the expressions for the maximum potential energy and the maximum kinetic energy, an expression for the natural frequency of the plate can be obtained.

Hence $$\omega^2 = U_{max} \Big/ \frac{\rho h}{2g}\int_0^l\int_{y_1}^{y_2} W^2\,dy\,dx \qquad 4.11$$

(a) Calculation of the natural frequencies of the family with no nodal lines perpendicular to the fixed edge.

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with no nodal lines perpendicular to the fixed edge of the cantilevered, right-angled plates, investigated experimentally, the deflected form of the vibrating plate is assumed to be

$$W = A\,\varphi_m(x)\,\theta_o(y) \qquad 4.12$$

Substituting equation 4.12 in equation 4.11 the first seven modes of the family of the plates under consideration were calculated. The calculated and experimental frequencies are shown in Table 5.

The angle θ , which is used to express the variation in area of the right-angled plates is shown in Figure 4.2

Table 5: . . .

VARIATION OF NATURAL FREQUENCY OF MODES 1/0 AND 2/0 WITH AREA OF PLATE

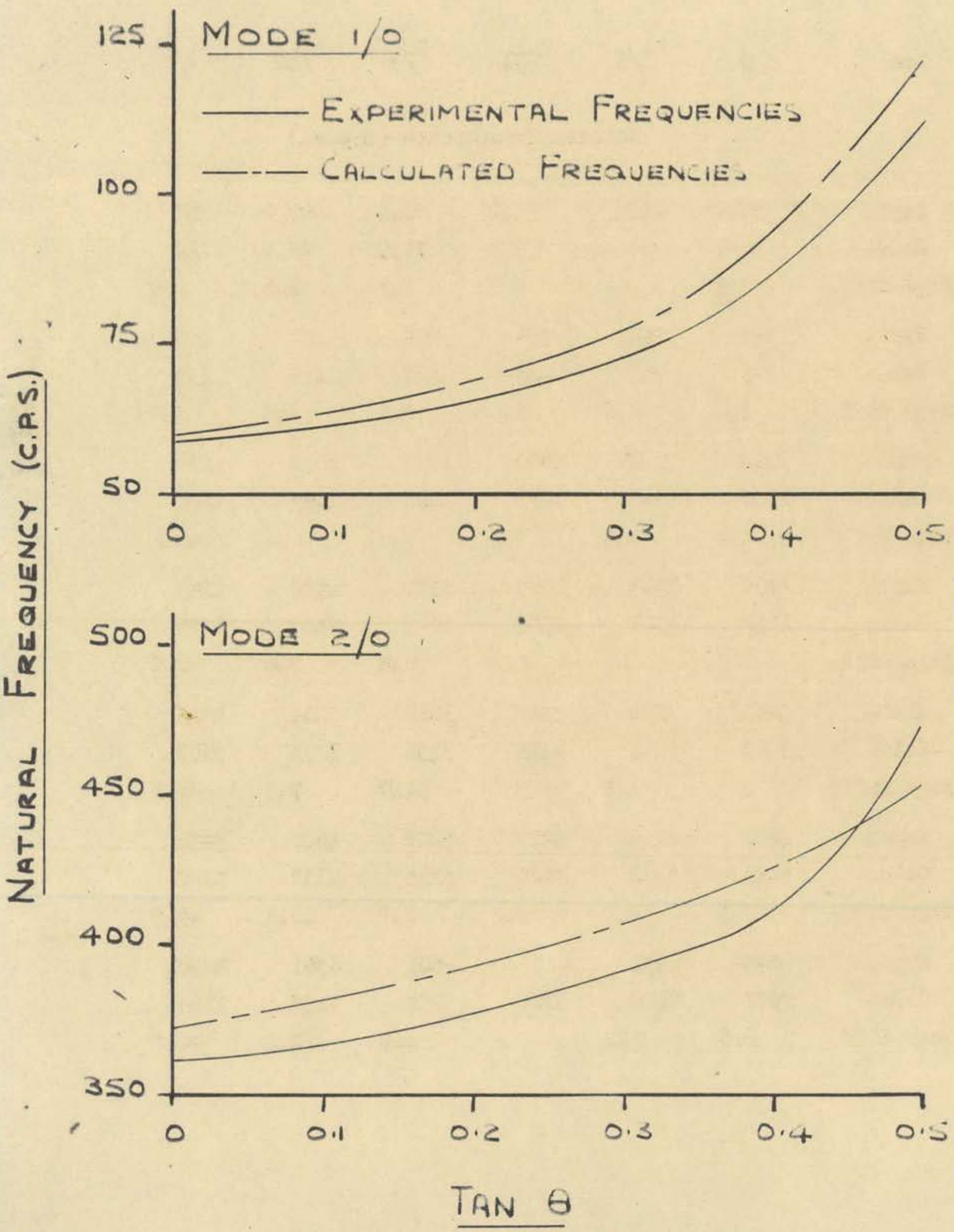


Table 5: Experimental and Calculated Natural Frequencies of the family m/0 of Cantilevered Right-angled Plates.

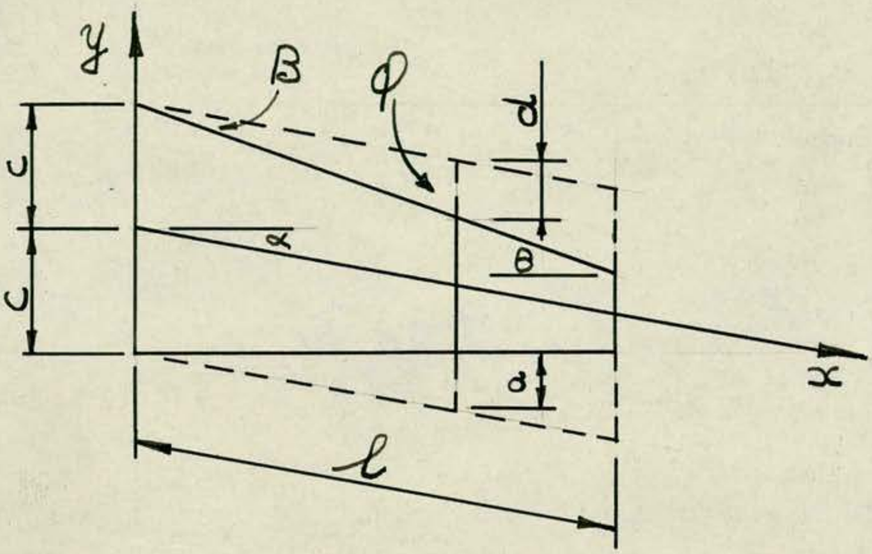
Assumed Deflected Form for Calculated Frequencies:-

$$W = A\phi_m(x)\,\theta_0(y)$$

	tan θ	0	$^1/_8$	1.$^1/_4$	$^1/_3$	$^5/_{12}$	$^1/_2$
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/0	Expt.	57.9	62.4	71.6	76.4	87.9	112
	Calc.	59.6	65.1	75.7	81.9	95.5	122
	%age diff.	2.9	4.3	5.7	7.2	8.6	8.9
2/0	Expt.	362	368	376	392	409	477
	Calc.	375	385	403	412	429	453
	%age diff.	3.6	4.6	7.1	5.1	4.9	-5.0
3/0	Expt.	1006	1015	1000	1013	1032	1157
	Calc.	1050	1060	1076	1084	1099	1120
	%age diff.	4.4	4.4	7.6	7.0	6.4	-3.2
4/0	Expt.	1969	1967	1905	1923	1959	2185
	Calc.	2058	2067	2084	2092	2107	2127
	%age diff.	4.5	5.1	9.4	8.8	7.6	-2.7
5/0	Expt.	3251	3239	3327	3067	3164	3559
	Calc.	3402	3411	3428	3436	3451	3471
	%age diff.	4.6	5.3	3.0	12.0	9.1	-2.5
6/0	Expt.	4817	4855	4857	5019	4660	5250
	Calc.	5082	5092	5108	5116	5131	5150
	%age diff.	5.5	4.9	5.2	1.9	10.1	-1.9
7/0	Expt.	6695	6691		6801	6361	7220
	Calc.	7098	7108	7124	7132	7147	7166
	%age diff.	6.0	6.2		4.9	12.4	-0.7

(b) Calculation of the natural Frequencies of the family with one nodal line "perpendicular" to the fixed edge.

Figure 4.3



The nodal patterns corresponding to the natural frequencies of right-angled trapezoidal and triangular plates, tend to be parallel to the x and y directions shown in Figure 4.3. The experimental results show that the nodal line in the x direction of the family with one nodal line in that direction coincides approximately with the x axis. To calculate the natural frequencies of that family, using the Rayleigh-Ritz method, the deflected form of the vibrating plate was represented in terms of the skew co-ordinates x and y.

The maximum potential energy of the vibrating plate in terms of the skew co-ordinates x and y is given by the expression,

$$U_{max} = \frac{D}{2\cos\alpha}\int_0^l\int_{y_1}^{y_2}\left\{\frac{1}{\cos^2\alpha}\left[\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}-2\sin\alpha\frac{\partial^2 W}{\partial x\partial y}\right]^2 -2(1-\mu)\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}-\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right]\right\}dy\,dx \qquad 4.13$$

and the maximum kinetic energy by

$$T_{max} = \frac{\rho h\omega^2\cos\alpha}{2g}\int_0^l\int_{y_1}^{y_2} W^2\,dy\,dx \qquad 4.14$$

where $y_1 = (a-c) = (x \sin\alpha - c)$

and $y_2 = (c-d) = (c - \frac{x \sin B}{\sin\varphi})$

Equating the expressions for the maximum potential energy and the maximum kinetic energy, an expression for the natural frequency of the plate is obtained.

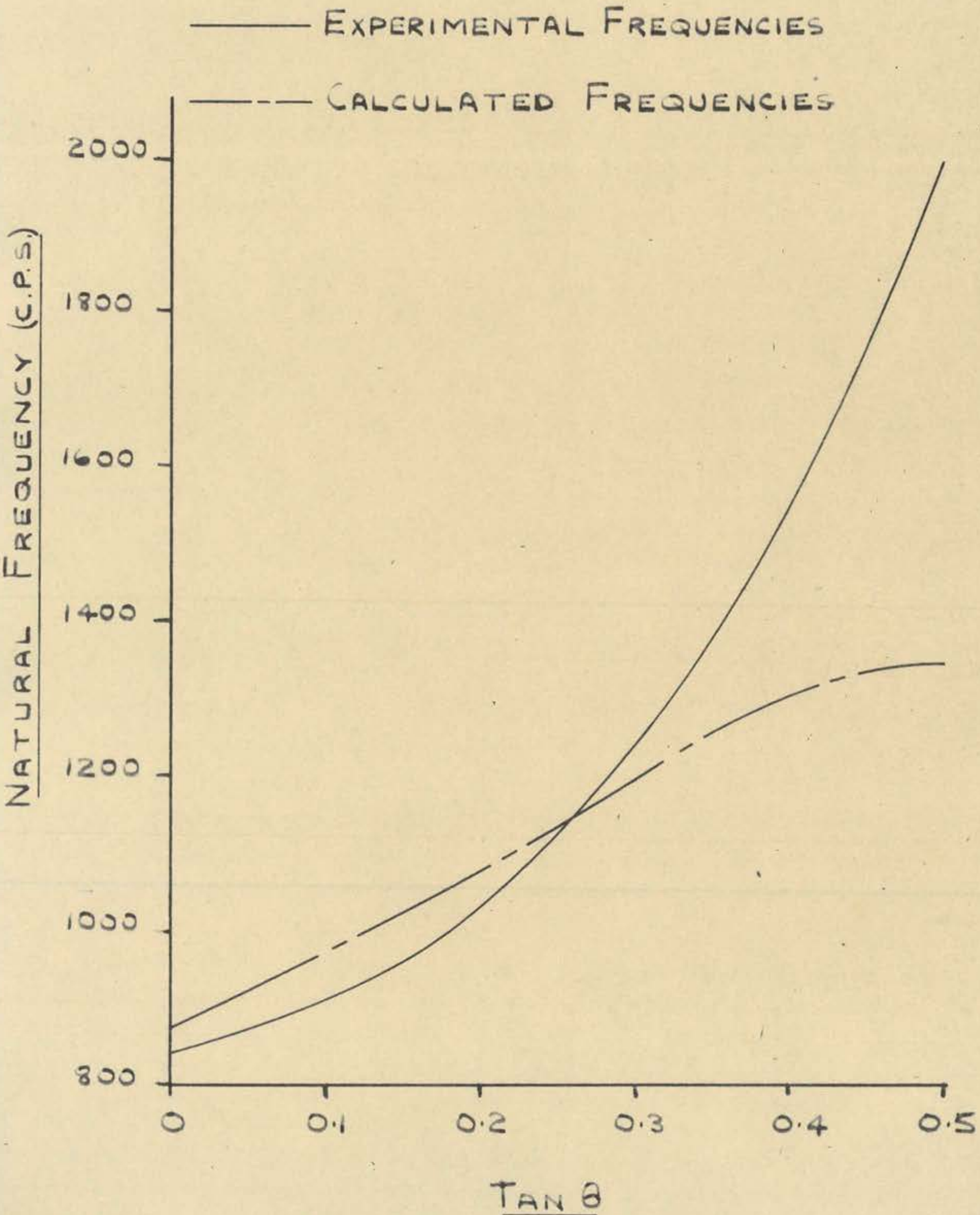
Hence $$\omega^2 = U_{max} \Big/ \rho\frac{h\cos\alpha}{2g}\int_0^l\int_{y_1}^{y_2} W^2 \, dy\, dx \qquad 4.15$$

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with one nodal line in the x direction, of the cantilevered right-angled plates investigated experimentally, the deflected form of the vibrating plate was assumed to be,

$$W = A\,\varphi_m(x)\,\theta_1(y) \qquad 4.16$$

Equation 4.16 was substituted in equation 4.15 and the natural frequencies of the first four modes of the family of the plates under consideration were calculated. The experimental and calculated frequencies are shown in Table 6.

VARIATION OF NATURAL FREQUENCY OF MODE 2/1 WITH AREA OF PLATE



VARIATION IN NATURAL FREQUENCY OF MODE 1/1 WITH AREA OF PLATE

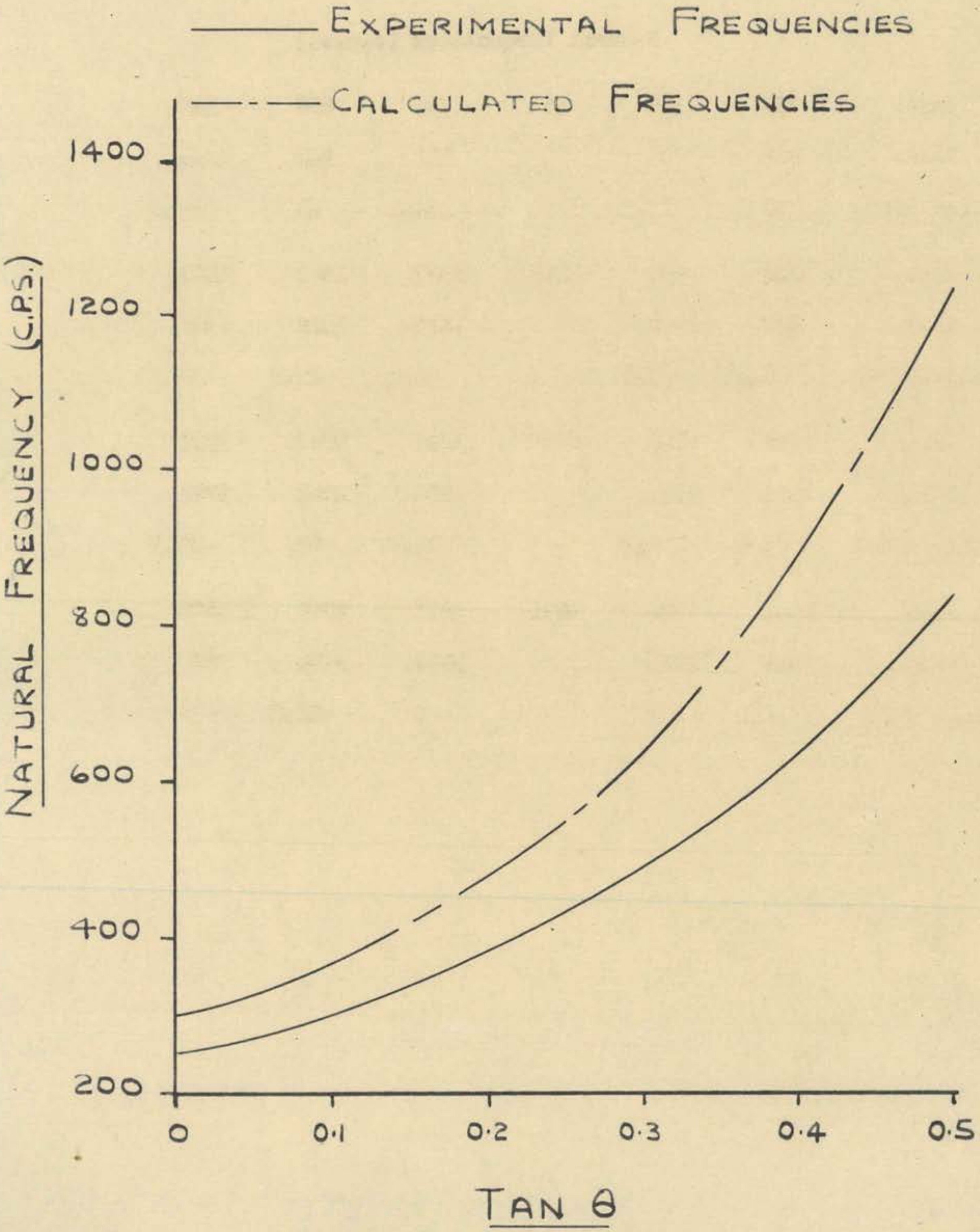


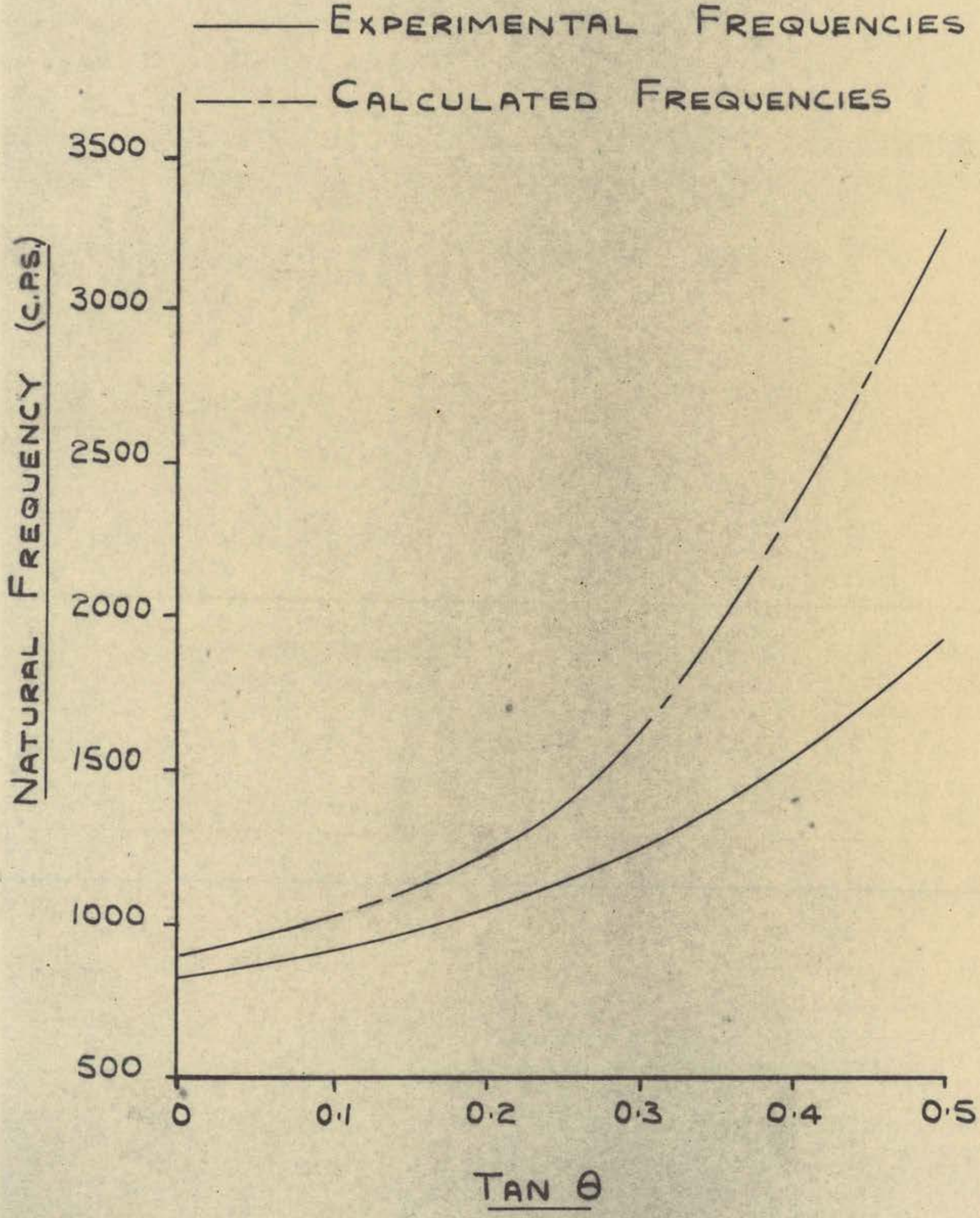
Table 6: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered Right-angled Plates.

Assumed deflected form for calculated frequencies:-

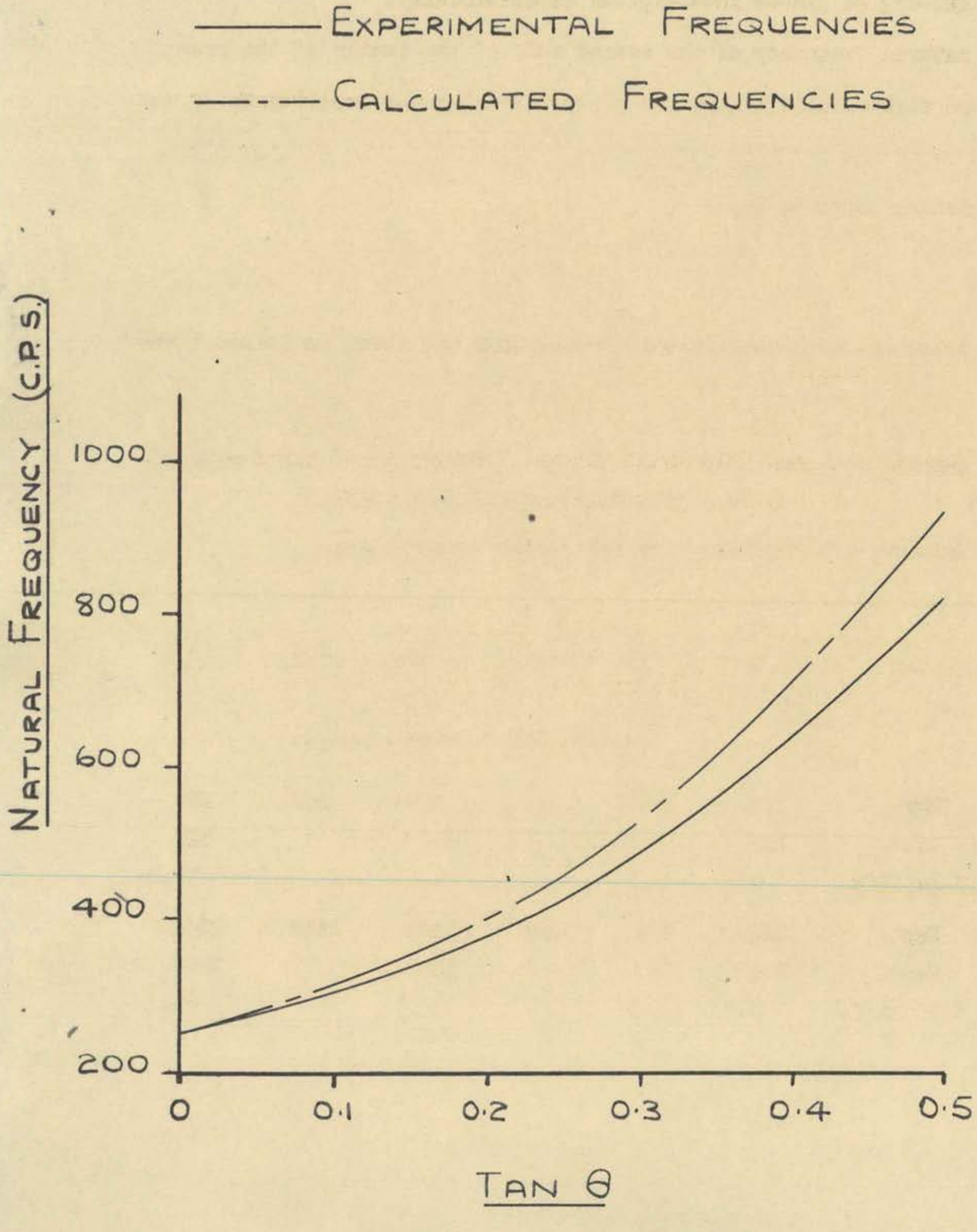
$$W = A\,\phi_m(x)\,\theta_1(y)$$

	$\tan\theta$	0	$^1/_8$	$1.^1/_4$	$^1/_3$	$^5/_{12}$	$^1/_2$
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/1	Expt.	255	319	461	535	668	820
	Calc.	306	394		712	942	1234
	%age diff.	20	23.5		34.1	41	50.4
2/1	Expt.	824	930	1185	1323	1598	1924
	Calc.	877	1000		1239	1318	1343
	%age diff.	6.4	7.5		-6.3	-17.5	-30.2
3/1	Expt.	1558	1714	2112	2291	2684	3137
	Calc.	1614	1754		2000	2067	2081
	%age diff.	3.6	2.3		-12.7	-23	-33.7
4/1	Expt.	2544	2724	3034	3515	4029	4670
	Calc.	2648	2807		3092	3176	3207
	%age diff.	4.1	3		-12	-21.2	-31.3

VARIATION OF NATURAL FREQUENCY OF MODE 2/1
WITH AREA OF PLATE



Variation of Natural Frequency of Mode 1/1 with Area of Plate



The Rayleigh-Ritz method was then used, assuming the deflected form of the vibrating plate to be

$$W = [A\phi_1(x) + B\phi_2(x)]\,\theta_1(y) \qquad 4.17$$

to calculate the natural frequencies of the modes 1/1 and 2/1 of the cantilevered right-angled plates investigated experimentally.

The natural frequency of the second mode of the family of the rectangular and right-angled triangular plates investigated experimentally was calculated using the Rayleigh-Ritz method, and assuming the deflected form of the vibrating plate to be,

$$W = [A\phi_2(x) + B\phi_3(x)]\,\theta_1(y) \qquad 4.18$$

The calculated and experimental frequencies are shown in Tables 7 and 8.

Table 7: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered Right-angled Plates.

Assumed deflected form for calculated frequencies:-

$$W = [A\phi_1(x) + B\phi_2(x)]\,\theta_1(y)$$

	$\tan\theta$	0	$^1/_8$	$1.^1/_4$	$^1/_3$	$^5/_{12}$	$^1/_2$
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/1	Expt. Calc. %age diff.	255 255 0	319 328 2.8	461	535 565 5.6	668	820 928 13.2
2/1	Expt. Calc. %age diff.	824 894 8.5	930 1055 13.5	1185	1323 1671 26.3	1598	1924 3260 69.4

Table 8: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Rectangular and Right-angled Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = [A\varphi_2(x) + B\varphi_3(x)]\,\theta_1(y)$$

		Rectangular Plate	Triangular Plate
Nodal Pattern	Natural Frequencies (c.p.s.)		
2/1	Expt. Calc. %age diff.	824 820 -0.5	1924 1312 -31.8

On examination of the results shown in Tables 1 to 8, it can be seen that the calculated values of the natural frequencies of mode 1/0 of all plate shapes are higher than the experimental values. In the case of the higher modes of the family, the calculated frequencies of the rectangular and trapezoidal shaped plates are higher than the experimental frequencies, but the calculated frequencies of the triangular plates are, in general, lower than the experimental frequencies. This is due to the fact, that the experimental frequencies of the higher modes of the family tend to be constant as the area of the plate is reduced, and then increase rapdily as the shape of the plate approaches a triangle. The rapid increase in frequency was not, however, predicted when the Rayleigh-Ritz method was used to calculate the natural frequencies of the higher modes of the family, assuming the deflected form of the vibrating plate to be a single term combination of the functions which represent the modes of vibration of uniform beams. As a result the calculated frequencies of these modes of the triangular plates are lower than the experimental frequencies.

The results show that, for all plate shapes, the calculated fre-

frequencies of mode 1/1 are higher than the experimental frequencies. The discrepancy between the calculated and experimental results varies from 19% for the rectangular plate to 38% for the triangular plate in the case of the symmetrical plates, and from 20% for the rectangular plate to 50.4% for the triangular plate in the case of the right-angled plates, when the frequencies are calculated assuming the deflected form of the vibrating plate to be

$$W = A\phi_1(x)\,\theta_1(y)$$

The discrepancy between the calculated and experimental results is, however, considerably reduced for all plate shapes, when the frequencies are calculated assuming the deflected form of the vibrating plate to be

$$W = \left[A\phi_1(x) + B\phi_2(x)\right]\theta_1(y)$$

The results of the higher modes of the family m/1 show that the increase in frequency, with the reduction in area of the plate, is less rapid for the calculated frequencies than for the experimental frequencies when the deflected form of the vibrating plate is assumed to be,

$$W = A\phi_m(x)\,\theta_1(y)$$

For example, the experimental frequency of mode 2/1 of the isosceles triangular plate is 30.2% higher than the calculated frequency. No improvement was obtained when the natural frequency of mode 2/1, of the cantilevered symmetrical and the cantilevered right-angled triangular plates was calculated assuming the deflected form of the vibrating plate to be,

$$W = \left[A\phi_2(x) + B\phi_3(x)\right]\theta_1(y)$$

The calculated value of the natural frequency of mode 2/1 is, however, higher than the experimental value, for all plate shapes, when, for the purposes of the calculations, the deflected form of the vibrating plate is assumed to be

$$W = \left[A\phi_1(x) + B\phi_2(x)\right]\theta_1(y)$$

The discrepancy between the calculated and experimental results varies from 8.7% for the rectangular plate to 50.6% for the triangular plate in the case of the symmetrical plates. The corresponding discrepancies of the right-angled plates are 8.5% and 69.4%

The comparison between the foregoing experimental and calculated results has shown, that the method used successfully by Warburton for the calculation of the natural frequencies of rectangular plates, is unsatisfactory when applied to cantilevered, trapezoidal and triangular shaped plates. It becomes necessary, therefore, to use some other assumed deflected form, if the Rayleigh-Ritz method is to prove successful for the calculation of the natural frequencies of cantilevered, trapezoidal and triangular shaped plates.

4.2 Series approximation of the deflected form of the vibrating plate.

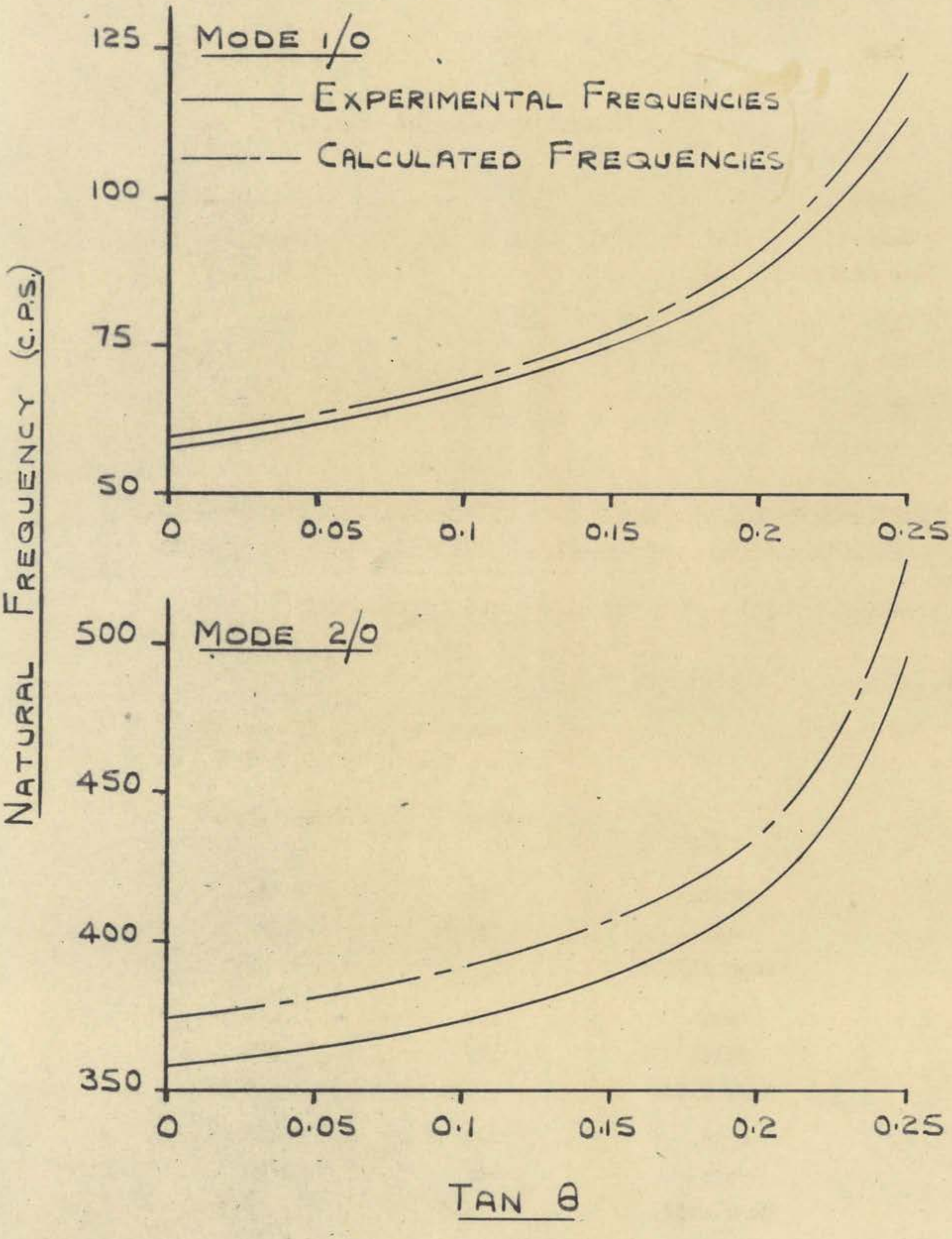
The results of the previous section have shown that the Reyleigh-Ritz method of calculating the natural frequencies of cantilevered, triangular and trapezoidal shaped plates is not, in general, satisfactory if the deflected form of the vibrating plate is assumed to be a single term combination of the appropriate functions which represent the normal modes of vibration of uniform beams. If the Rayleigh-Ritz method is to prove successful as a method of calculating the natural frequencies of these plate shapes, some other assumed deflected forms must be used. It was decided, therefore, to calculate their natural frequencies using a series consisting of the appropriate beam functions, to represent the deflected form of the vibrating plate. The natural frequencies of each family were again considered separately and for the calculation of the natural frequencies of the families with no nodal lines and one nodal line perpendicular to the fixed edge the shape of the deflected form was assumed respectively to be of the form

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + \cdots\cdots] \theta_0(y) \qquad 4.19$$

ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:-

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x)]\,\theta_0(y)$$

VARIATION OF NATURAL FREQUENCY OF MODES 1/0 AND 2/0 WITH AREA OF PLATE



$$W = [a_1 \varphi_1(x) + a_2 \varphi_2(x) + \cdots] \theta_1(y) \tag{4.20}$$

4.2.1. Natural Frequencies of Symmetrical Plates.

(a) Calculation of the natural frequencies of the family with no nodal lines perpendicular to the fixed edge.

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with no nodal lines perpendicular to the fixed edge of the plate of the cantilevered, symmetrical plates investigated experimentally, the deflected form of the vibrating plate was assumed in turn to be,

$$W = [a_1 \varphi_1(x) + a_2 \varphi_2(x)] \theta_0(y) \tag{4.21}$$

$$W = [a_1 \varphi_1(x) + a_2 \varphi_2(x) + a_3 \varphi_3(x)] \theta_0(y) \tag{4.22}$$

$$W = [a_1 \varphi_1(x) + a_2 \varphi_2(x) + a_3 \varphi_3(x) + a_4 \varphi_4(x)] \theta_0(y) \tag{4.23}$$

Using the first of these expressions, the first two natural frequencies of the family of the cantilevered, rectangular, symmetrical, trapezoidal and the isosceles triangular plates were calculated. With the deflected form assumed, respectively, to be equations 4.22 and 4.23, the first three and the first four natural frequencies of the family of the rectangular and isosceles triangular plates, were calculated. The calculated and experimental results are shown in Tables 9, 10 and 11.

Table 9: . . .

Table 9: Experimental and Calculated Natural Frequencies of the family m/O of Cantilevered Symmetrical Plates.

Assumed deflected form for calculated frequencies:-

$$W = [a_1 \varphi_1(x) + a_2 \varphi_2(x)] \theta_0(y)$$

	tan θ	0	$^1/_{16}$	$^1/_8$	$^3/_{16}$	$^5/_{24}$	$^1/_4$
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/0	Expt.	58	63.4	70.2	83.4	90.1	114
	Calc.	59.5	64.9	73	87	94.7	121
	%age diff.	2.6	1.7	4	4.3	5.1	6.1
2/0	Expt.	359	369	380	409	422	494
	Calc.	373	383	400	426	443	528
	%age diff.	3.9	3.8	5.3	4.2	5.0	6.9

Table 10: Experimental and Calculated Natural Frequencies of the family m/O of Cantilevered, Rectangular and Isosceles Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = [a_1 \varphi_1(x) + a_2 \varphi_2(x) + a_3 \varphi_3(x)] \theta_0(y)$$

		Rectangular Plate	Triangular Plate
Nodal Pattern	Natural Frequencies (c.p.s.)		
1/0	Expt.	58	114
	Calc.	59.5	121
	%age diff.	2.6	6.1
2/0	Expt.	359	494
	Calc.	373	526
	%age diff.	3.9	6.5
3/0	Expt.	1003	1193
	Calc.	1046	1283
	%age diff.	4.3	7.5

Table 11: Experimental and Calculated Natural Frequencies of the family m/O of Cantilevered, Rectangular and Isosceles Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = [a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + a_4\varphi_4(x)]\,\theta_0(y)$$

		Rectangular Plate	Triangular Plate
Nodal Pattern		Natural Frequencies (c.p.s.)	
1/0	Expt.	58	114
	Calc.	59.5	121
	%age diff.	2.6	6.1
2/0	Expt.	359	494
	Calc.	373	526
	%age diff.	3.9	6.5
3/0	Expt.	1003	1193
	Calc.	1046	1280
	%age diff.	4.3	7.3
4/0	Expt.	1970	2202
	Calc.	2049	2371
	%age diff.	4	7.7

(b) Calculation of the natural frequencies of the family with one nodal line perpendicular to the fixed edge.

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with one nodal line perpendicular to the fixed edge of the cantilevered, symmetrical plates, investigated experimentally, the deflected form of the vibrating plate was assumed, in turn, to be

$$W = [a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x)]\,\theta_1(y) \qquad 4.24$$

$$W = [a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + a_4\varphi_4(x)]\,\theta_1(y) \qquad 4.25$$

Using equation 4.24 to represent the deflected form of the vibrating plate, the first three natural frequencies of the family of the rectangular plate and the first two natural frequencies of the family of the isosceles triangular plate were calculated. With the deflected form assumed to be equation 4.25, the first four natural frequencies of the family of the rectangular plate were calculated and the first three natural frequencies of the family of the isosceles triangular plate were calculated. The calculated and experimental results are shown in Tables 12 and 13.

Table 12: Experimental and Calculated Natural frequencies of the family m/1 of Cantilevered, Rectangular and Isosceles Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x)]\, \theta_1(y)$$

		Rectangular Plate	Triangular Plate
Nodal Pattern	Natural Frequencies (c.p.s.)		
1/1	Expt.	256	821
	Calc.	254	861
	%age diff.	-0.8	4.9
2/1	Expt.	820	1994
	Calc.	832	2128
	%age diff.	1.5	6.7
3/1	Expt.	1558	
	Calc.	1639	
	%age diff.	5.2	

Table 13: . . .

Table 13: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Rectangular and Isosceles Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x) + a_4 \phi_4(x)]\, \theta_1(y)$$

		Rectangular Plate	Triangular Plate
Nodal Pattern		Natural Frequencies (c.p.s.)	
1/1	Expt.	256	821
	Calc.	252	861
	%age diff.	-1.6	4.9
2/1	Expt.	820	1994
	Calc.	831	2137
	%age diff.	1.3	7.2
3/1	Expt.	1558	3416
	Calc.	1601	3679
	%age diff.	2.8	7.7
4/1	Expt.	2545	
	Calc.	2651	
	%age diff.	4.2	

4.2.2 Natural Frequencies of Right-angled Plates

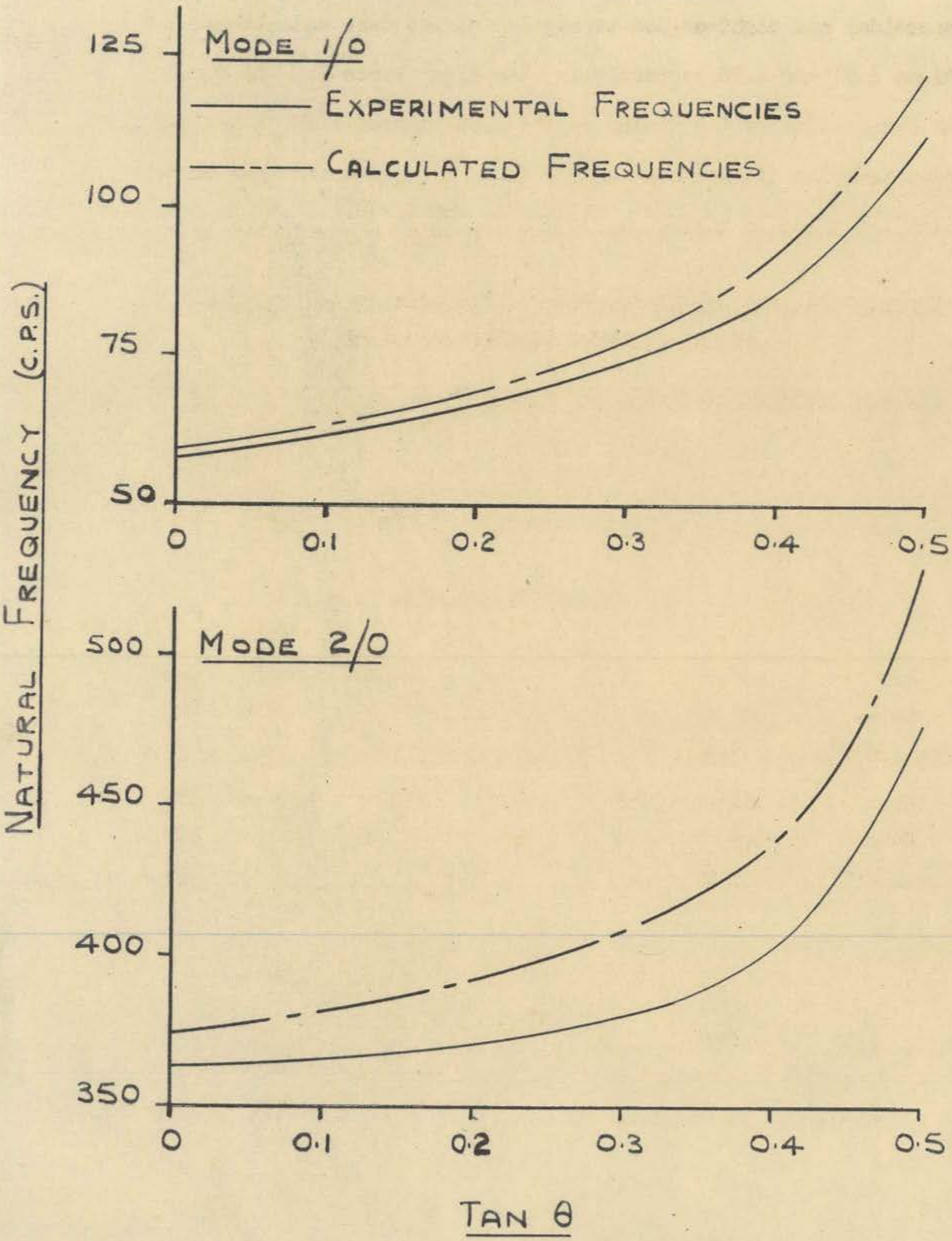
(a) Calculation of the natural frequencies of the family with no nodal lines perpendicular to the fixed edge.

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with no nodal lines perpendicular to the fixed edge of the cantilevered, right-angled plates investigated experimentally, the deflected form of the vibrating plate was assumed, in turn, to be

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x)]\, \theta_0(y) \qquad 4.26$$

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x)]\, \theta_0(y) \qquad 4.27$$

VARIATION OF NATURAL FREQUENCY OF MODES 1/0 AND 2/0 WITH AREA OF PLATE



$$W = [a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + a_4\varphi_4(x)]\theta_0(y) \qquad 4.28$$

With the deflected form assumed to be equation 4.26, the first two natural frequencies of the family of the cantilevered, rectangular, right-angled, trapezoidal and right-angled triangular plates were calculated. Using equations 4.27 and 4.28 respectively, the first three and the first four natural frequencies of the family of the cantilevered rectangular and right-angled triangular plates were calculated. The calculated and experimental results are shown in tables 14, 15 and 16.

Table 14: Experimental and Calculated Natural Frequencies of the family m/0 of Cantilevered Right-angled Plates.

Assumed deflected form for calculated frequencies:-

$$W = [a_1\varphi_1(x) + a_2\varphi_2(x)]\theta_0(y)$$

	$\tan\theta$	0	$1/8$	$1.1/4$	$1/3$	$5/12$	$1/2$
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/0	Expt.	57.9	62.4	71.6	76.4	87.9	112
	Calc.	59.6	65.1		81.6	95.1	122
	%age diff.	2.9	4.3		6.8	8.2	8.9
2/0	Expt.	362	368	376	392	409	477
	Calc.	375	385		417	445	530
	%age diff.	3.6	4.6		6.4	8.1	11.1

Table 15: . . .

Table 16: Experimental and Calculated Natural Frequencies of the family m/0 of Cantilevered, Rectangular and Right-angled, Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x) + a_4 \phi_4(x)]\, \theta_0(y)$$

		Rectangular Plate	Triangular Plate
Nodal Pattern		Natural Frequencies (c.p.s.)	
1/0	Expt.	57.9	112
	Calc.	59.6	122
	%age diff.	2.9	8.9
2/0	Expt.	362	477
	Calc.	375	528
	%age diff.	3.6	10.7
3/0	Expt.	1006	1157
	Calc.	1050	1285
	%age diff.	4.4	11.1
4/0	Expt.	1969	2185
	Calc.	2058	2380
	%age diff.	4.5	8.9

(b) Calculation of the natural frequencies of the family with one nodal line "perpendicular" to the fixed edge.

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with one nodal line in the x direction, (figure 4.3) of the cantilevered, right-angled plates investigated experimentally, the deflected form of the vibrating plate was assumed to be,

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x)]\, \theta_1(y) \qquad 4.29$$

With the deflected form assumed to be equation 4.29 the first three natural frequencies of the family of the cantilevered, rectangular plate and the first two natural frequencies of the family of the cantilevered, right-angled triangular plate were calculated. The calculated and experimental frequencies are shown in Table 17.

Table 17: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Rectangular and Right-angled Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = [a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x)]\theta_1(y)$$

		Rectangular Plate	Triangular Plate
Nodal Pattern		Natural Frequencies (c.p.s.)	
1/1	Expt.	255	820
	Calc.	255	918
	%age diff.	0	12.0
2/1	Expt.	824	1924
	Calc.	835	2275
	%age diff.	1.3	18.2
3/1	Expt.	1558	
	Calc.	1645	
	%age diff.	5.6	

It can be seen from the results that the use of the Rayleigh-Ritz method to calculate the natural frequencies of cantilevered, rectangular, trapezoidal and triangular shaped plates, assuming the deflected form of the vibrating plate to be a series consisting of the appropriate beam functions, is, in general, satisfactory. Apart from mode 1/1 of the rectangular plates, the calculated frequencies of all modes considered, of all plate shapes, are higher than the experimental frequencies. For the family m/0 of the symmetrical plates, the discrepancy between the calculated and experimental frequencies varies from 0.8% to 7.7%. In the case of the right-angled plates, the discrepancy between the calculated and experimental frequencies of the family m/0 varies from 2.9% to 11.3% and the discrepancy between the calculated and experimental frequencies of the family m/1 varies from 0% to 18.2%.

The use of a series of beam functions to represent the deflected form of cantilevered, trapezoidal and triangular shaped plates leads, however, to the solution of integrals of the type $\int_0^l x\,\phi_1(x)\,\phi_2(x)\,dx$, when the expression for the deflected form is substituted in the expressions for the maximum potential energy and the maximum kinetic energy of the vibrating plate. The solution of these integrals is extremely difficult without the use of an electronic digital computer. They were solved, therefore, with the aid of a Ferranti Pegasus Computer, and the results obtained are tabulated in Appendix No. 2.

CHAPTER V

Calculation of the natural frequencies of cantilevered plates using algebraic Functions to represent the deflected form of the vibrating plate.

When it was found that theRayleigh-Ritz method, using a single term combination of the appropriate beam functions to represent the deflected form of the vibrating plate was unsatisfactory for the calculation of the natural frequencies of cantilevered, trapezoidal and triangular shaped plates, the use of two alternative types of deflected form were investigated. These were a series consisting of the appropriate beam functions and a series of algebraic functions. It can be seen from the work of the previous chapter that satisfactory results are obtained for the natural frequencies of the families with no nodal lines and one nodal line perpendicular to the fixed edge of cantilevered, rectangular, trapezoidal and triangular plates, if the deflected form of the vibrating plate is assumed to be a series of beam functions. In this chapter, the possibility of using a series of algebraic functions to represent the deflected form of the vibrating plate will be investigated.

To calculate the natural frequencies of the families with no nodal lines and one nodal line perpendicular to the fixed edge, respectively of cantilevered plates, the deflected form of the vibrating plate was assumed to be of the form

$$W = (a_1 x^2 + a_2 x^3 + ---) \qquad 5.1$$

$$W = (a_1 x^2 + a_2 x^3 + ---)y \qquad 5.2$$

As in the previous chapter thesymmetrical and the right-angled plates will be considered separately and the natural frequencies of the plates will be divided into families.

5.1 Natural Frequencies of Symmetrical Plates

(a) Calculation of the natural frequencies of the family with no nodal lines perpendicular to the fixed edge.

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with no nodal lines perpendicular to the fixed edge of the cantilevered, symmetrical plates, investigated experimentally, the deflected form of the vibrating plate was assumed, in turn, to be

$$W = a_1 x^2 + a_2 x^3 \tag{5.3}$$

$$W = a_1 x^2 + a_2 x^3 + a_3 x^4 \tag{5.4}$$

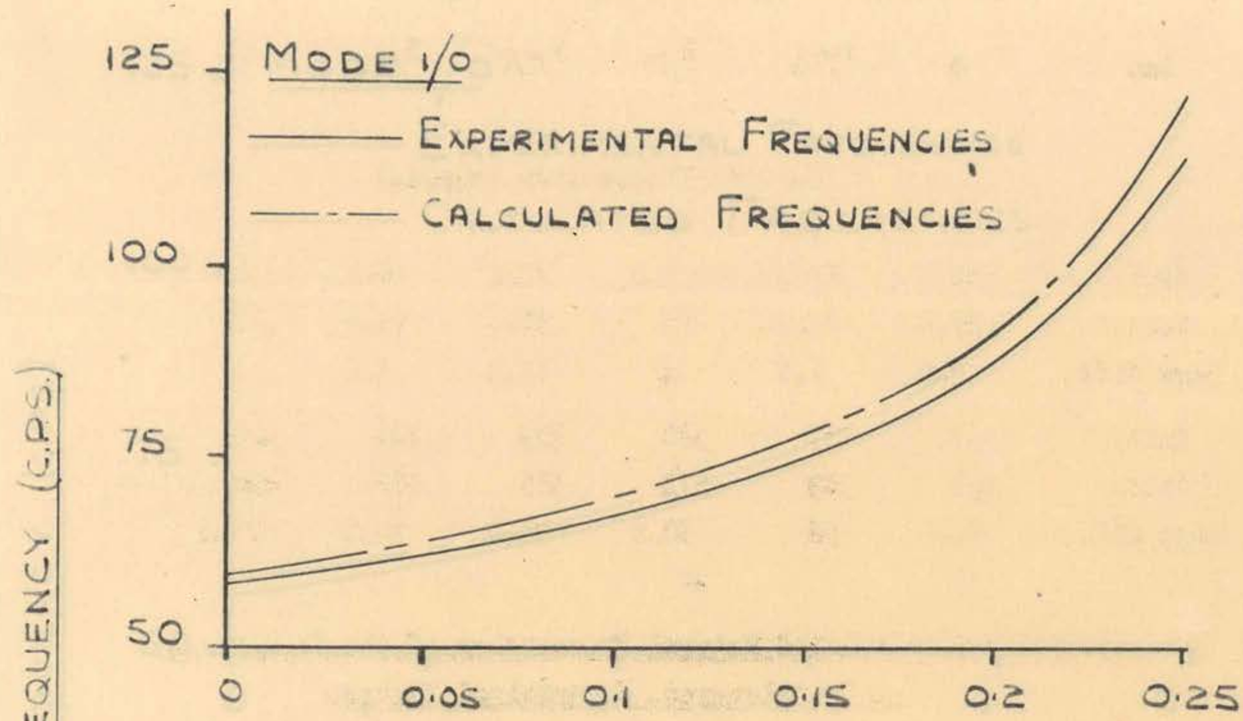
$$W = a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 \tag{5.5}$$

Using equations 5.3 and 5.4, the first two natural frequencies of the family of the cantilevered, symmetrical plates were calculated and using equation 5.5 the first three natural frequencies of the family of the cantilevered, rectangular and the cantilevered isosceles triangular plates were calculated. The calculated and experimental results are shown in Tables 18, 19 and 20.

ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES :-

$$W = a_1 x^2 + a_2 x^3$$

VARIATION OF NATURAL FREQUENCY OF MODES 1/0 AND 2/0 WITH AREA OF PLATE



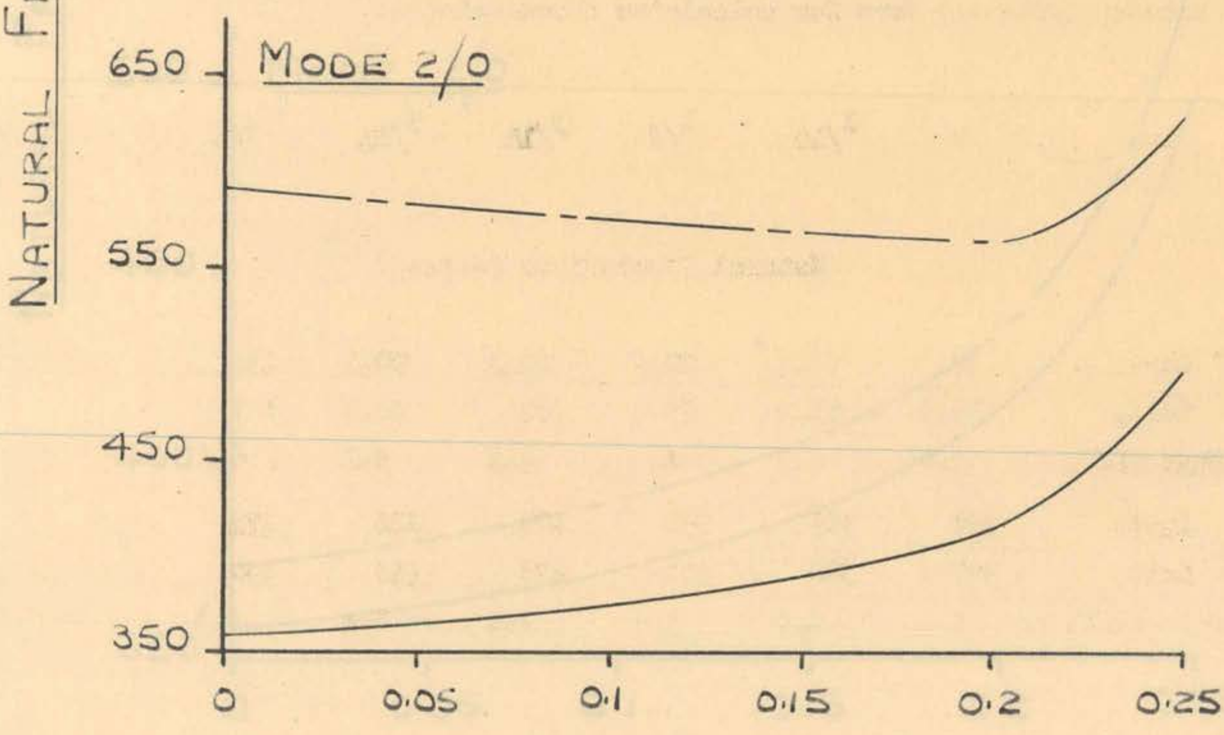


Table 18: Experimental and Calculated Natural Frequencies of the family m/O of Cantilevered Symmetrical Plates.

Assumed deflected form for calculated frequencies:-

$$W = a_1 x^2 + a_2 x^3$$

	tan θ	0	1/16	1/8	3/16	5/24	1/4
Nodal Patterns	Natural Frequencies (c.p.s.)						
1/0	Expt.	58	63.4	70.2	83.4	90.1	114
	Calc.	59.6	64.9	73	87	94.7	122
	%age diff.	2.8	1.7	4	4.3	5.1	7
2/0	Expt.	359	369	380	409	422	494
	Calc.	590	583	573	565	566	630
	%age diff.	64.5	58	50.8	38.2	34.1	27.6

Table 19: Experimental and Calculated Natural Frequencies of the family m/O of Cantilevered, Symmetrical Plates.

Assumed deflected form for calculated frequencies:-

$$W = a_1 x^2 + a_2 x^3 + a_3 x^4$$

	tan θ	0	1/16	1/8	3/16	5/24	1/4
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/0	Expt.	58	63.4	70.2	83.4	90.1	114
	Calc.	59.5	64.9	73	87	94.7	121
	%age diff.	2.6	1.7	4	4.3	5.1	6.1
2/0	Expt.	359	369	380	409	422	494
	Calc.	377	386	401	428	445	530
	%age diff.	5	4.6	5.5	4.4	5.4	7.3

Table 20: Experimental and Calculated Natural Frequenices of the family m/O of Cantilevered, Rectangular and Isosceles Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$$

		Rectangular Plate	Triangular Plate
Nodal Pattern		Natural Frequencies (c.p.s.)	
1/0	Expt.	58	114
	Calc.	59.5	121
	%age diff.	2.6	6.1
2/0	Expt.	359	494
	Calc.	376	528
	%age diff.	4.7	6.9
3/0	Expt.	1003	1193
	Calc.	1074	1315
	%age diff.	6.9	10.2

(b) Calculation of the natural frequencies of the family with one nodal line perpendicular to the fixed edge.

With the deflected form of the vibrating plate assumed, in turn, to be

$$W = (a_1x^2 + a_2x^3)y \qquad 5.6$$

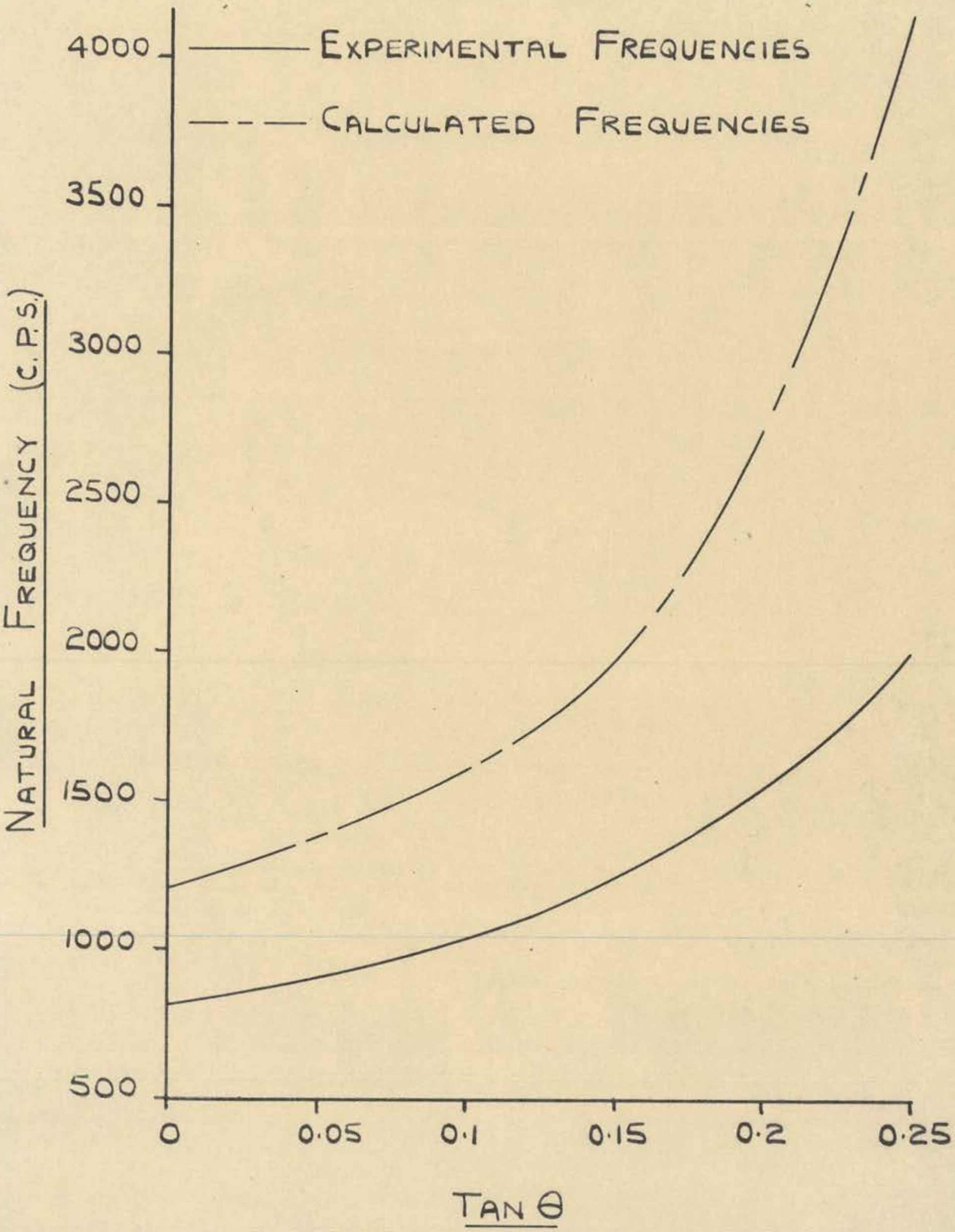
$$W = (a_1x^2 + a_2x^3)y + a_3x^2y^3 \qquad 5.7$$

$$W = (a_1x^2 + a_2x^3 + a_3x^4)y \qquad 5.8$$

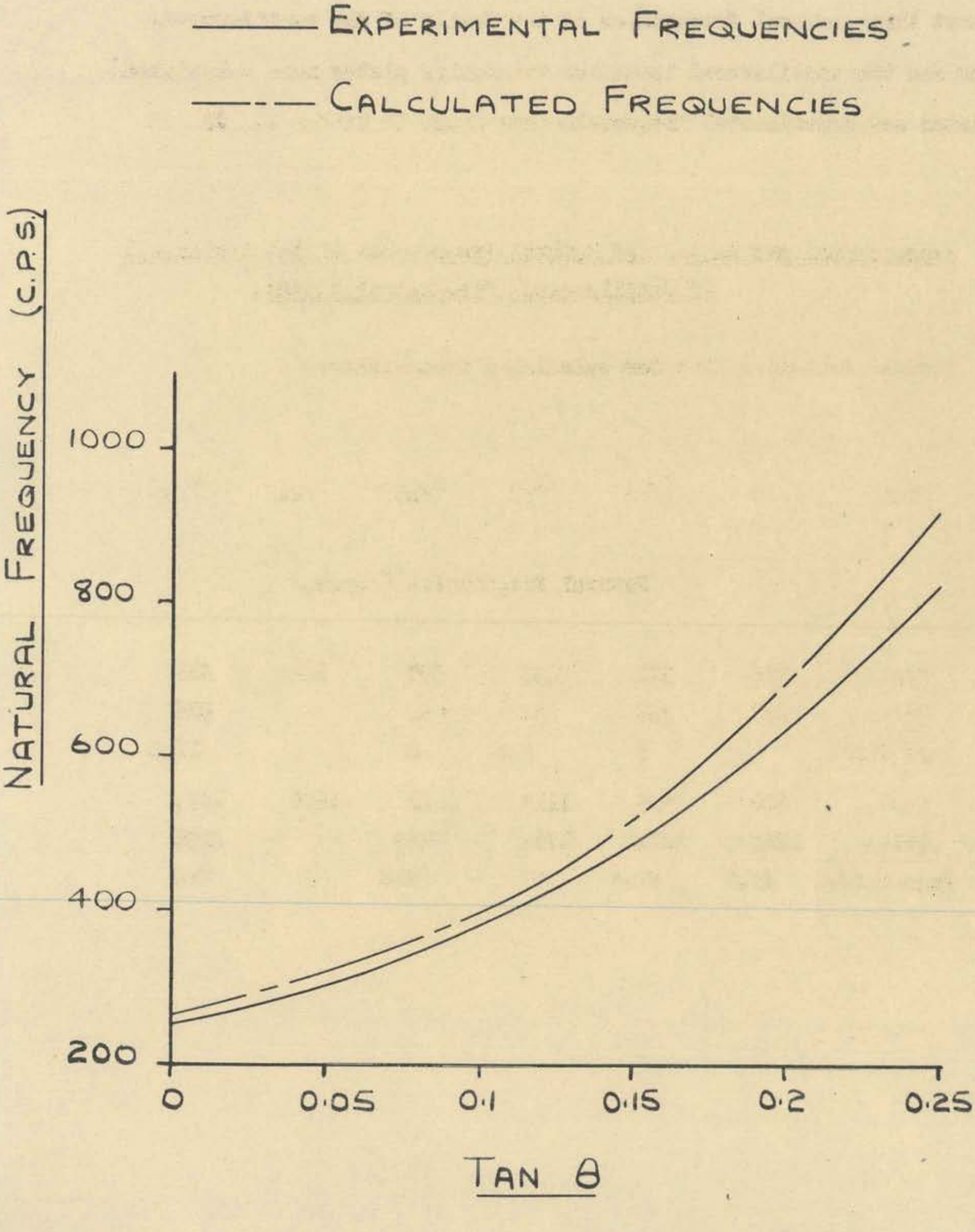
$$W = (a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5)y \qquad 5.9$$

the Rayleigh-Ritz method was used to calculate the natural frequencies of the

VARIATION OF NATURAL FREQUENCY OF MODE 2/1 WITH AREA OF PLATE



Variation of Natural Frequency of Mode 1/1 with Area of Plate



family with one nodal line perpendicular to the fixed edge of the cantilevered symmetrical plates investigated experimentally.

Using equations 5.6 and 5.7, the first two natural frequencies of the family of these plates were calculated. With the deflected form assumed to be equations 5.8 and 5.9 respectively, the first two natural frequencies and the first three natural frequencies of the family of the cantilevered, rectangular and the cantilevered isosceles triangular plates were calculated. The calculated and experimental frequencies are shown in Tables 21, 22, 23 and 24.

Table 21: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Symmetrical Plates.

Assumed deflected form for calculated frequencies:-

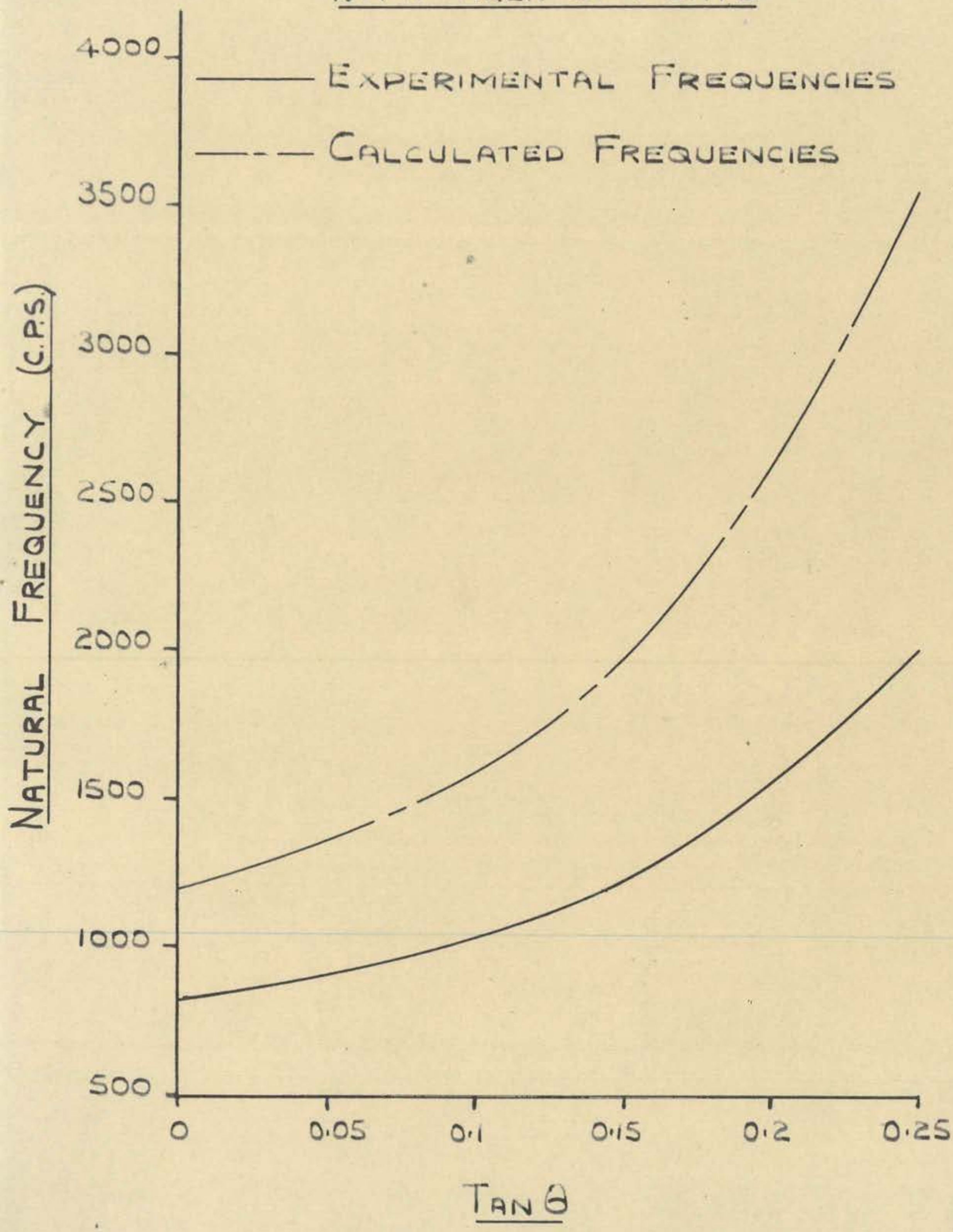
$$W = (a_1 x^2 + a_2 x^3) y$$

	$\tan\theta$	0	$^1/_{16}$	$^1/_8$	$^3/_{16}$	$^5/_{24}$	$^1/_4$
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/1	Expt.	256	322	430	597	666	821
	Calc.	268	338	452	645		918
	%age diff.	4.7	5	5.1	8		11.8
2/1	Expt.	820	939	1119	1446	1606	1994
	Calc.	1227	1422	1761	2499		4136
	%age diff.	49.7	51.5	57.4	72.8		112

ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:

$$W = (a_1 x^2 + a_2 x^3)y + a_3 x^2 y^3$$

VARIATION OF NATURAL FREQUENCY OF MODE 2/1 WITH AREA OF PLATE



ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:-

$$W = (a_1 x^2 + a_2 x^3) y + a_3 x^2 y^3$$

VARIATION OF NATURAL FREQUENCY OF MODE 1/1 WITH AREA OF PLATE

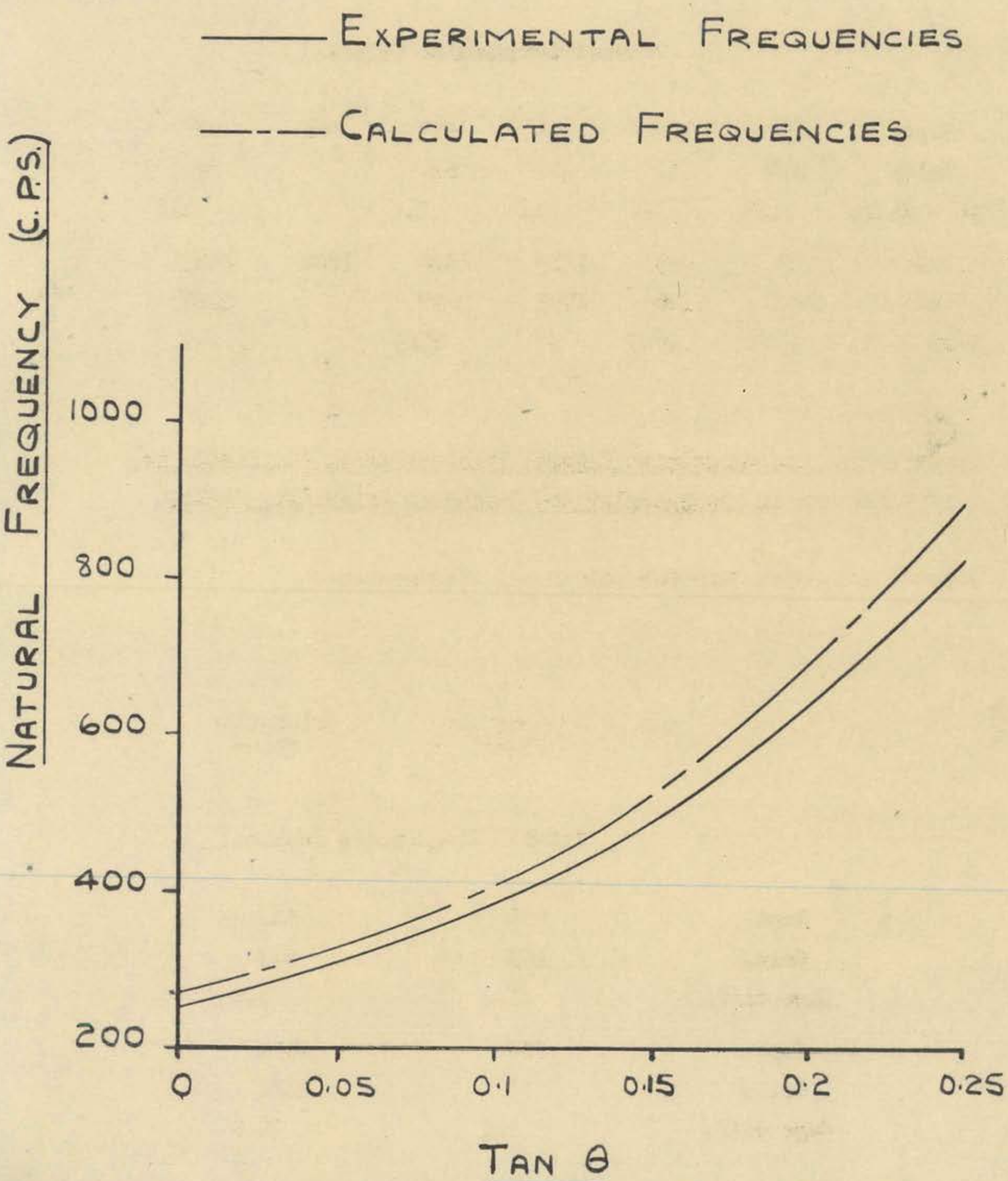


Table 22: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Symmetrical Plates.

Assumed deflected form for calculated frequencies:-

$$W = (a_1 x^2 + a_2 x^3) y + a_3 x^2 y^3$$

	$\tan\theta$	0	1/16	1/8	3/16	5/24	1/4
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/1	Expt.	256	322	430	597	666	821
	Calc.	268	341	452	640		893
	%age diff.	4.7	5.9	5.1	7.2		8.8
2/1	Expt.	820	939	1119	1446	1606	1994
	Calc.	1204	1406	1755	2423		4050
	%age diff.	46.8	49.7	57	67.9		108

Table 23: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Rectangular and Isosceles Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = (a_1 x^2 + a_2 x^3 + a_3 x^4) y$$

		Rectangular Plate	Triangular Plate
Nodal Pattern	Natural Frequencies (c.p.s.)		
1/1	Expt.	256	821
	Calc.	253	863
	%age diff.	-1.2	5.1
2/1	Expt.	820	1994
	Calc.	856	2432
	%age diff.	4.4	22.0

Table 24: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Rectangular and Isosceles Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = (a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5)y$$

		Rectangular Plate	Triangular Plate
Nodal Pattern	Natural Frequencies (c.p.s.)		
1/1	Expt.	256	821
	Calc.	257	859
	%age diff.	0.4	4.6
2/1	Expt.	820	1994
	Calc.	856	2142
	%age diff.	4.4	7.4
3/1	Expt.	1558	3416
	Calc.	1842	4248
	%age diff.	18.3	24.4

5.2 Natural Frequencies of Right-angled Plates

(a) Calculation of the natural frequencies of the family with no nodal lines perpendicular to the fixed edge.

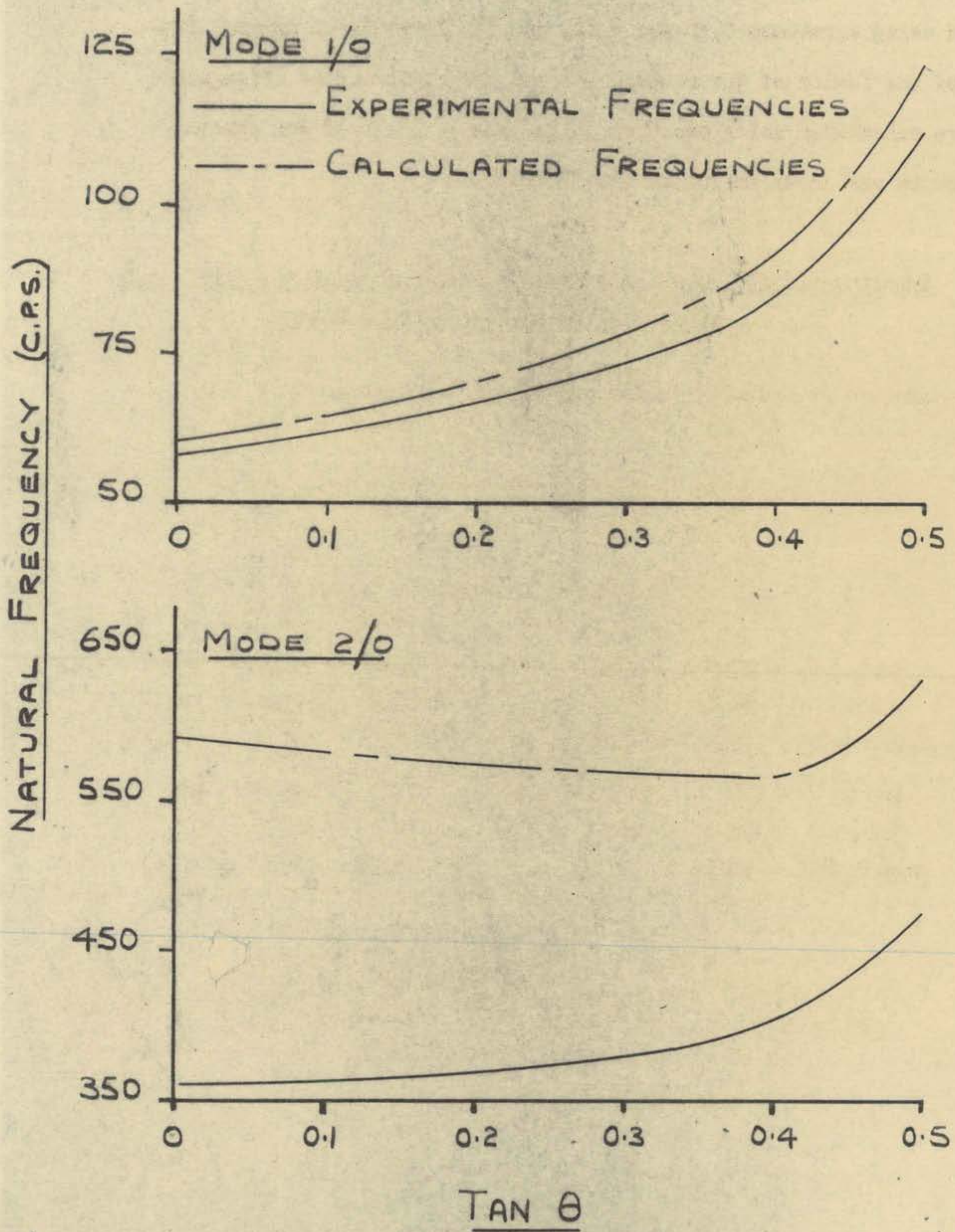
The Rayleigh-Ritz method, assuming the deflected form of the vibrating plate to be, in turn,

$$W = a_1x^2 + a_2x^3 \qquad 5.10$$

$$W = a_1x^2 + a_2x^3 + a_3x^4 \qquad 5.11$$

$$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 \qquad 5.12$$

Variation of Natural Frequency of Modes 1/0 and 2/0 with Area of Plate



was used to calculate the natural frequencies of the family with no nodal lines perpendicular to the fixed edge of the cantilevered, right- angled plates investigated experimentally.

The first two natural frequencies of the family of these plates were calculated using equations 5.10 and 5.11, and the first three natural frequencies of the family of the rectangular and the right-angled triangular plates were calculated using equation 5.12. The calculated and experimental results are shown in Tables 25, 26 and 27.

Table 25: Experimental and Calculated Natural Frequencies of the family m/0 of Cantilevered, Right-angled Plates.

Assumed deflected form for calculated frequencies:-

$$W = a_1 x^2 + a_2 x^3$$

	tan θ	0	1/8	1.1/4	1/3	5/12	1/2
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/0	Expt. Calc. %age diff.	57.9 59.6 2.9	62.4 65.4 4.8	71.6	76.4 8 82.2 7.6	87.9 95.1 8.2	112 123 9.8
2/0	Expt. Calc. %age diff.	362 592 62.4	368 585 59.3	376	392 569 45.2	409 568 38.8	477 632 32.3

ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:-

$$W = a_1 x^2 + a_2 x^3 + a_3 x^4$$

VARIATION OF NATURAL FREQUENCY OF MODES 1/0 AND 2/0 WITH AREA OF PLATE

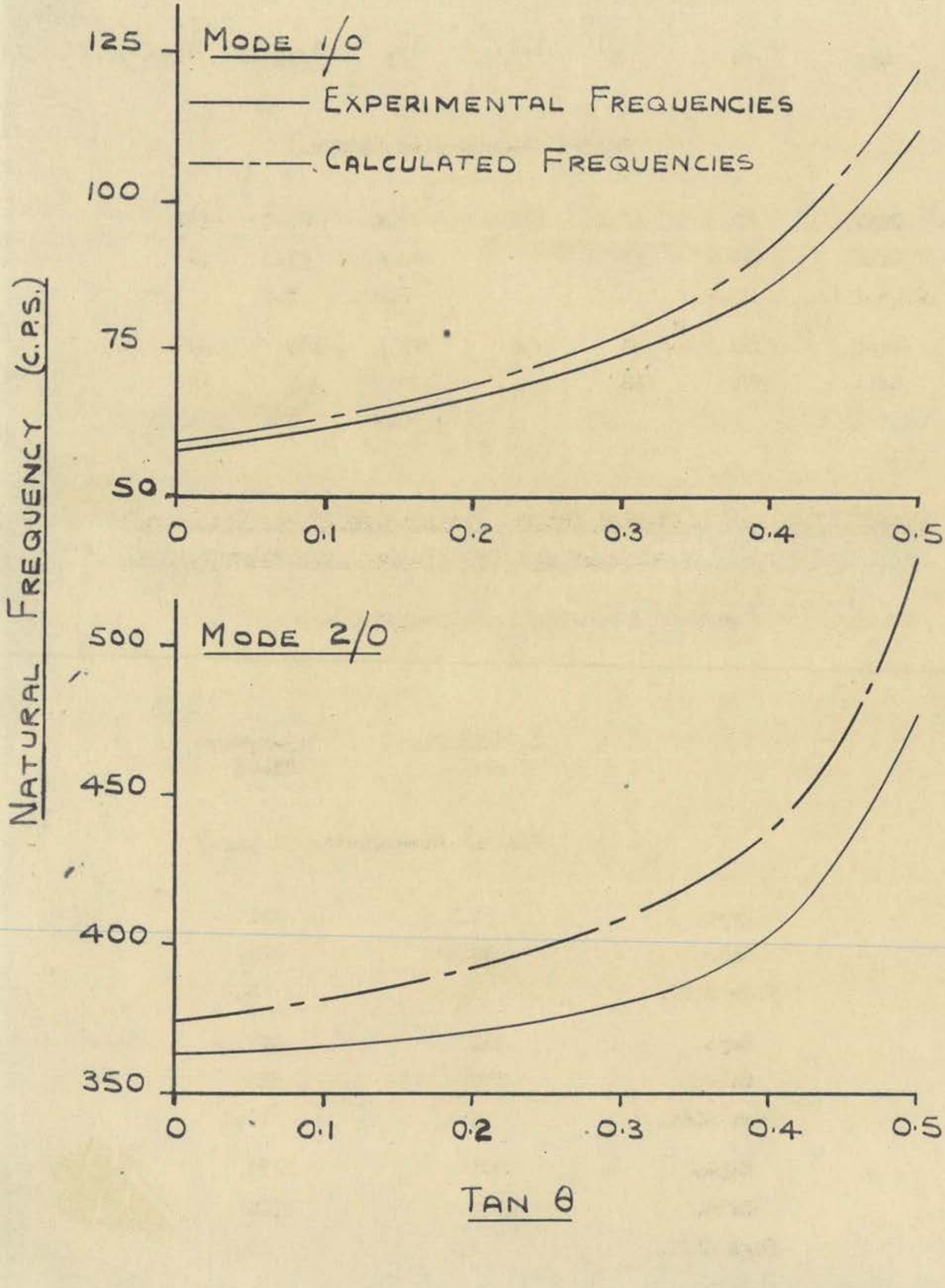


Table 26: Experimental and Calculated Natural Frequencies of the family m/O of Cantilevered, Right-angled Plates.

Assumed deflected form for calculated frequencies:-

$$W = a_1 x^2 + a_2 x^3 + a_3 x^4$$

	$\tan\theta$	0	$^1/_8$	$1.^1/_4$	$^1/_3$	$^5/_{12}$	$^1/_2$
Nodal Pattern	Natural Frequencies (c.p.s.)						
1/0	Expt.	57.9	62.4	71.6	76.4	87.9	112
	Calc.	59.6	65.1		81.6	95.1	122
	%age diff.	2.9	4.3		6.8	8.2	8.9
2/0	Expt.	362	368	376	392	409	477
	Calc.	378	388		419	447	532
	%age diff.	4.4	5.4		6.9	9.3	11.5

Table 27: Experimental and Calculated Natural Frequencies of the family m/O of Cantilevered, Rectangular and Right-angled Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5$$

		Rectangular Plate	Triangular Plate
Nodal Pattern	Natural Frequencies (c.p.s.)		
1/0	Expt.	57.9	112
	Calc.	59.5	122
	%age diff.	2.8	8.9
2/0	Expt.	362	477
	Calc.	377	530
	%age diff.	4.1	11.1
3/0	Expt.	1006	1157
	Calc.	1078	1320
	%age diff.	7.2	14.1

(b) Calculation of the natural frequencies of the family with one nodal line "perpendicular" to the fixed edge.

Using the Rayleigh-Ritx method to calculate the natural frequencies of the family with one nodal line in the x direction (Figure 4.3) of the cantilevered, right-angled plates investigated experimentally, the deflected form of the vibrating plate was assumed, in turn, to be,

$$W = (a_1 x^2 + a_2 x^3) y \tag{5.13}$$

$$W = (a_1 x^2 + a_2 x^3 + a_3 x^4) y \tag{5.14}$$

Using equations 5.13 and 5.14 the first two natural frequencies of the family of the cantilevered rectangular and the cantilevered, right-angled triangular plates were calculated. The calculated and experimental results are shown in Tables 28 and 29.

Table 28: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Rectangular and Right-angled, Triangular Plates

Assumed deflected form for calculated frequencies:-

$$W = (a_1 x^2 + a_2 x^3) y$$

		Rectangular Plate	Triangular Plate
Nodal Pattern		Natural Frequencies (c.p.s.)	
1/1	Expt.	255	820
	Calc.	269	1048
	%age diff.	5.5	27.8
2/1	Expt.	824	1924
	Calc.	1230	4737
	%age diff.	49.3	146.2

Table 29: Experimental and Calculated Natural Frequencies of the family m/1 of Cantilevered, Rectangular and Right-angled Triangular Plates.

Assumed deflected form for calculated frequencies:-

$$W = (a_1 x^2 + a_2 x^3 + a_3 x^4)\, y$$

		Rectangular Plate	Triangular Plate
Nodal Pattern		Natural Frequencies (c.p.s.)	
1/1	Expt.	255	820
	Calc.	255	926
	%age diff.	0	12.9
2/1	Expt.	824	1924
	Calc.	859	2909
	%age diff.	4.2	51.2

It can be seen from the results that when the Rayleigh-Ritz method is used to calculate the natural frequencies of the families m/0 and m/1 of rectangular trapezoidal and triangular plates, assuming the deflected form of the vibrating plate to be a series of algebraic functions, the results obtained are, in general, satisfactory. It was found, for example, that when a four term series is used to calculate the natural frequencies of the family m/0 of rectangular, isosceles triangular and right-angled triangular plates, agreement between the experimental and calculated results is satisfactory, for the first three frequencies of the family, for each of these plates. The discrepancy between the calculated and experimental frequencies of the symmetrical plates varies from 2.6% for mode 1/0 to 6.9% for mode 3/0 in the case of the rectangular plate and from 6.1% for mode 1/0 to 10.2% for mode 3/0 in the case of the isosceles triangular plate. The corresponding discrepancies, when the four term series is used to calculate the natural frequencies of the family m/0 of the right-angled plates, are 2.8% for mode 1/0 and 7.2% for mode 3/0, in the case of the rectangular plate, and 8.9% for mode 1/0 and 14.1% for mode 3/0, in the case of the right-angled triangular plate.

Using a four term series to calculate the natural frequencies of the family m/1 of the symmetrical plates it was found that, for the rectangular and isosceles triangular plates, the calculated value of the natural frequency compared favourable with the experimental value for the first two modes of the family. The differences between the calculated and experimental results were 0.4% for mode 1/1 and 4.4% for mode 2/1, in the case of the rectangular plate, and 4.6% for mode 1/1 and 7.4% for mode 2/1 in the case of the isosceles triangular plate. The discrepancy between the calculated and experimental results for mode 3/1 was 18.3% for the rectangular plate and 24.4% for the isosceles triangular plate.

A three term series was used to calculate the natural frequencies of the family m/1 of the right-angled plates, and it was found that the discrepancies between the calculated and experimental frequencies were 0% for mode 1/1 and 4.2% for mode 2/1 in the case of the rectangular plate and 12.9% for mode 1/1 and 51.2% for mode 2/1 in the case of the right-angled triangular plate.

During the course of the investigations, the possibility of calculating the natural frequencies of the family m/2 of the symmetrical plates, using algebraic functions to represent the deflected form of the vibrating plate was investigated. The possibility of using an algebraic series of twelve terms to represent the deflected form of the vibrating plate for the calculation of the natural frequencies of the family m/1 of symmetrical plates was also investigated. The results of these investigations were, however, unsatisfactory. Details of the results and the series which were used to represent the deflected form of the vibrating plate are given in Appendices 5 and 6.

CHAPTER VI

ASSIMILATION OF RESULTS

In this chapter the results, which are shown in chapters two, four and five are presented for comparison in three tables. In the first the experimental results are presented to show the variation of experimental frequency with the angle of taper of the plate. In the second table, each mode is considered separately, and the results are presented to show for each assumed deflected form, the variation of frequency with the angle of taper of the plate. In the third table each plate shape is considered separately and the calculated results which were obtained for each mode of the plate are shown. The results are presented in the second and third tables to enable the frequencies, which were obtained from different deflected forms, of each mode to be compared. The experimental results of the modes, for which calculated frequencies have been obtained, are also included in the second and third tables.

The values of the non-dimensional frequency factor, λ, which was obtained for each mode from the various deflected forms, and the values of the coefficients in the deflected forms are also included in this chapter.

Using the expression

$$\lambda = \frac{\rho h \omega^2 l^4}{g D}$$

the non-dimensional frequency factors may be used to calculate the natural frequencies of cantilevered plates having the same angles of taper as those which were considered in the present research programme and having any dimensions which make the length to fixed edge ratio two to one.

The values of the coefficients a_m which were obtained from the series of beam functions of the form

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + - - -]\,\theta_0(y)$$

and

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + - - -]\,\theta_1(y)$$

(equations 4.19 and 4.20, pps. 46 & 47)

may be used for plates, having the same angles of taper as those which were considered in the present research programme, and any dimensions which make the length to fixed edge ratio two to one. If the coefficients, a_m, had been calculated for the algebraic series of the form,

$$W = a_1 x^2 + a_2 x^3 + ----$$

and (equations 5.1 & 5.2, p. 57)

$$W = (a_1 x^2 + a_2 x^3 + ---)y$$

they would have applied only to plates having the dimensions of those which were considered in the present research programme. In order that the algebraic series may be used for any plate having a length to fixed edge ratio of two to one, the series which were used to calculate the values of the coefficients, a_m, were of the form

$$W = a_1 \left(\frac{x}{l}\right)^2 + a_2 \left(\frac{x}{l}\right)^3 + -----$$

and

$$W = \left[a_1 \left(\frac{x}{l}\right)^2 + a_2 \left(\frac{x}{l}\right)^3 + ----\right] \frac{2y}{b}$$

6.1 VARIATION OF EXPERIMENTAL NATURAL FREQUENCIES WITH ANGLE OF TAPER

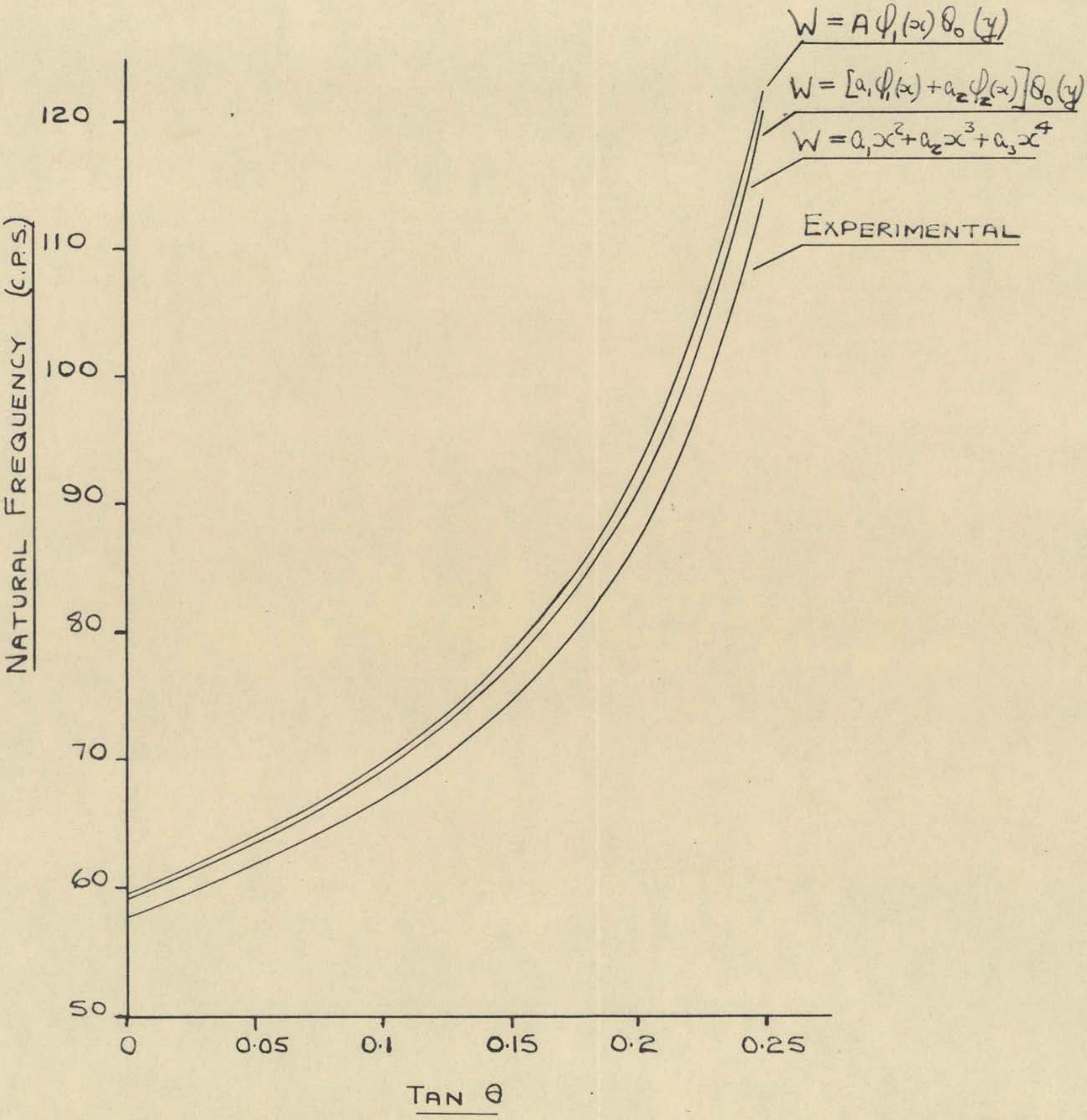
1) Symmetrical Plates

Tan θ	0	1/16	1/8	3/16	5/24	1/4
Mode	Natural Frequency (c.p.s.)					
1/0	58	63.4	70.2	83.4	90.1	114
2/0	359	369	380	409	422	494
3/0	1003	1011	1022	1046	1062	1193
4/0	1970	1966	1976	1996	2011	2202
5/0	3255	3237	3233		3250	3485
6/0	4823	4846	4757	4792	4807	5100
7/0	6690	6683	6668	6604	6632	6964
1/1	256	322	430	597	666	821
2/1	820	939	1119	1446	1606	1994
3/1	1558	1728	1994	2473	2722	3416
4/1	2545	2739	3079	3712	4056	5096
5/1	3808	4016	4395	5175	5607	7009
6/1	5353	5577	5968	6833		
7/1	7251	7417	7776			
1/2	1561	2192	2745	3277	3429	3785
2/2	2111	2725	3884	4987	5363	6111
3/2	2963	3581	4988	6780	7375	8553
4/2	4082	4706	6022		9515	
5/2	5435	6127	7414			
6/2	7041	7773	8987			
1/3	4143	5324				
2/3	4591	6394	6191			
3/3		7079	7992			
4/3	6599	8133				
5/3	8024					

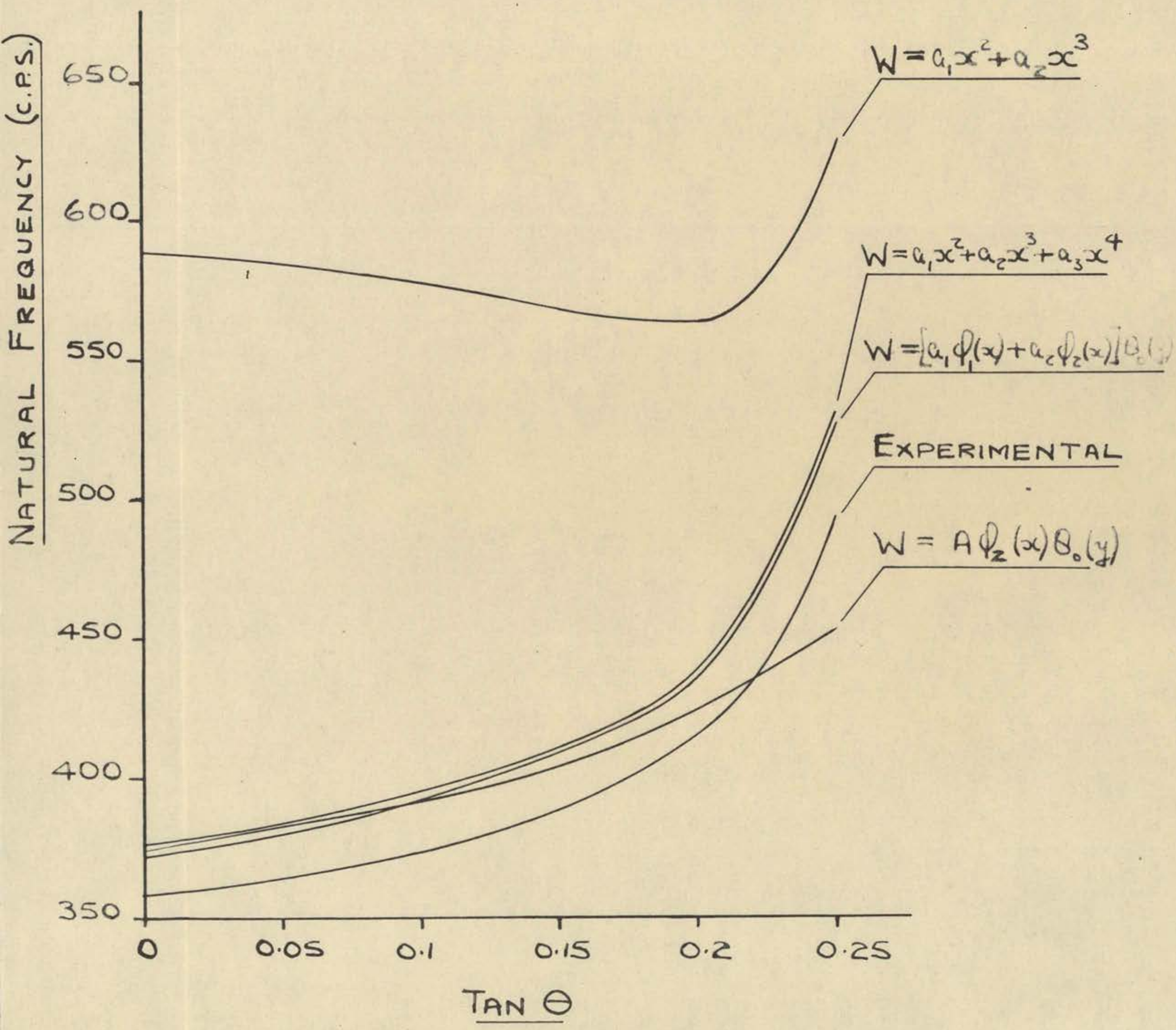
2) Right-angled Plates

Tan θ	0	1/8	1.1/4	1/3	5/12	1/2
Mode	Natural Frequency (c.p.s.)					
1/0	57.9	62.4	71.6	76.4	87.9	112
2/0	362	368	376	392	409	477
3/0	1006	1015	1000	1013	1032	1157
4/0	1969	1967	1905	1923	1959	2185
5/0	3251	3239	3327	3067	3164	3559
6/0	4817	4855	4857	5019	4660	5250
7/0	6695	6691		6801	6361	7220
1/1	255	319	461	535	668	820
2/1	824	930	1185	1323	1598	1924
3/1	1558	1714	2112	2291	2684	3137
4/1	2544	2724	3034	3515	4029	4670
5/1	3803	3988	4381	4461	5689	
6/1	5337	5519	5911	6002		
7/1	7235	7325	7422	8054		
1/2	1556	2195	2880	3168	3513	
2/2	2111	2729	4137	4706	5300	
3/2	2963	3584	5378	6306		
4/2	4079	4700		7796		
5/2	5449	6114	7970			
6/2	7053	7800	9440			
1/3	4141	5333				
2/3	4590	6316				
3/3		7061				
4/3	6608	8115				

Variation of Natural Frequency of Mode 1/0 of Symmetrical Plates with Angle of Taper



VARIATION OF NATURAL FREQUENCY OF MODE 2/0 OF SYMMETRICAL PLATES WITH ANGLE OF TAPER



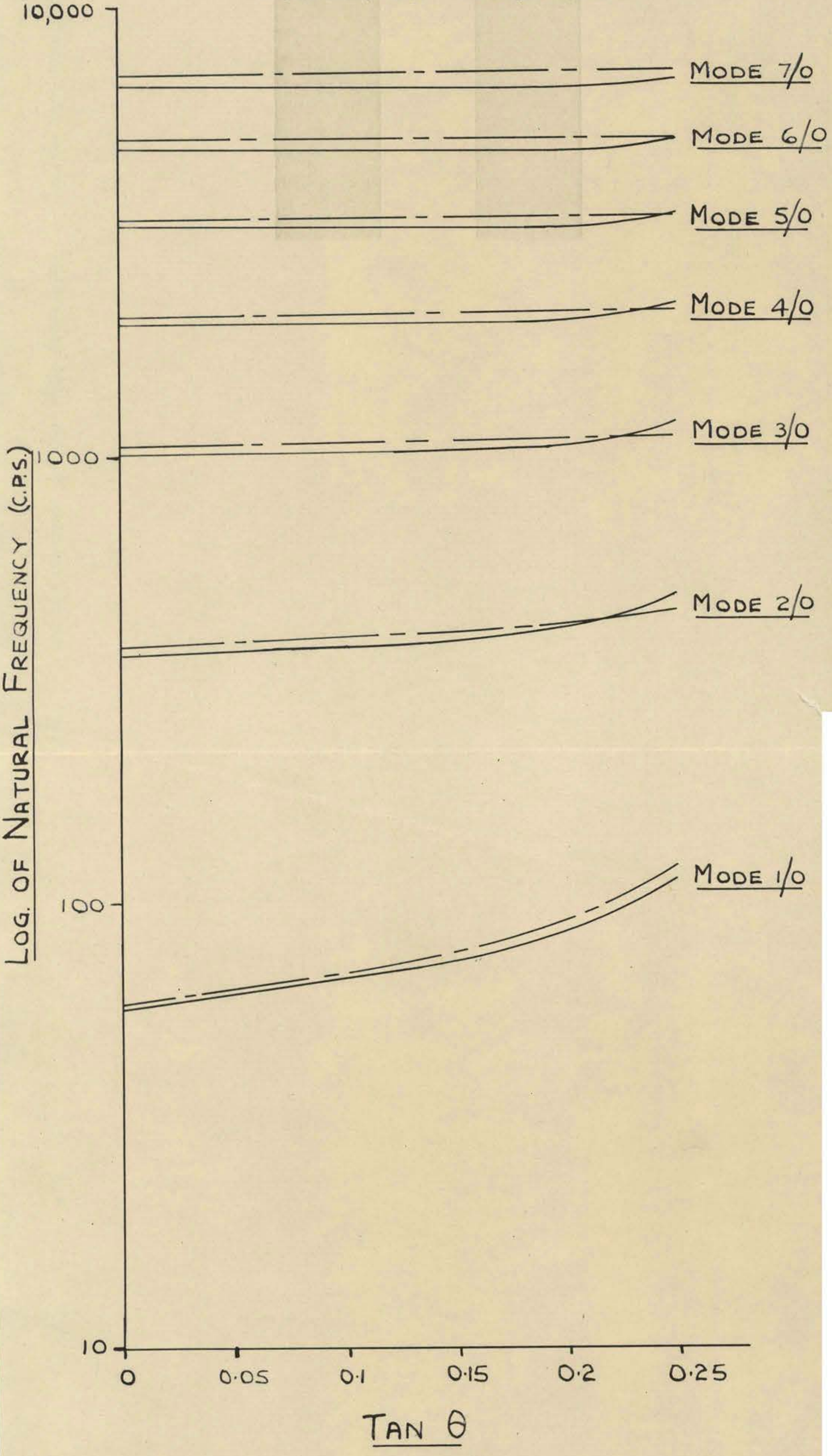
VARIATION OF NATURAL FREQUENCY OF MODES m/0 OF SYMMETRICAL PLATES WITH ANGLE OF TAPER

ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:-

$$W = A\,\phi_m(x)\,\theta_0(y)$$

——— EXPERIMENTAL FREQUENCIES

—·— CALCULATED FREQUENCIES



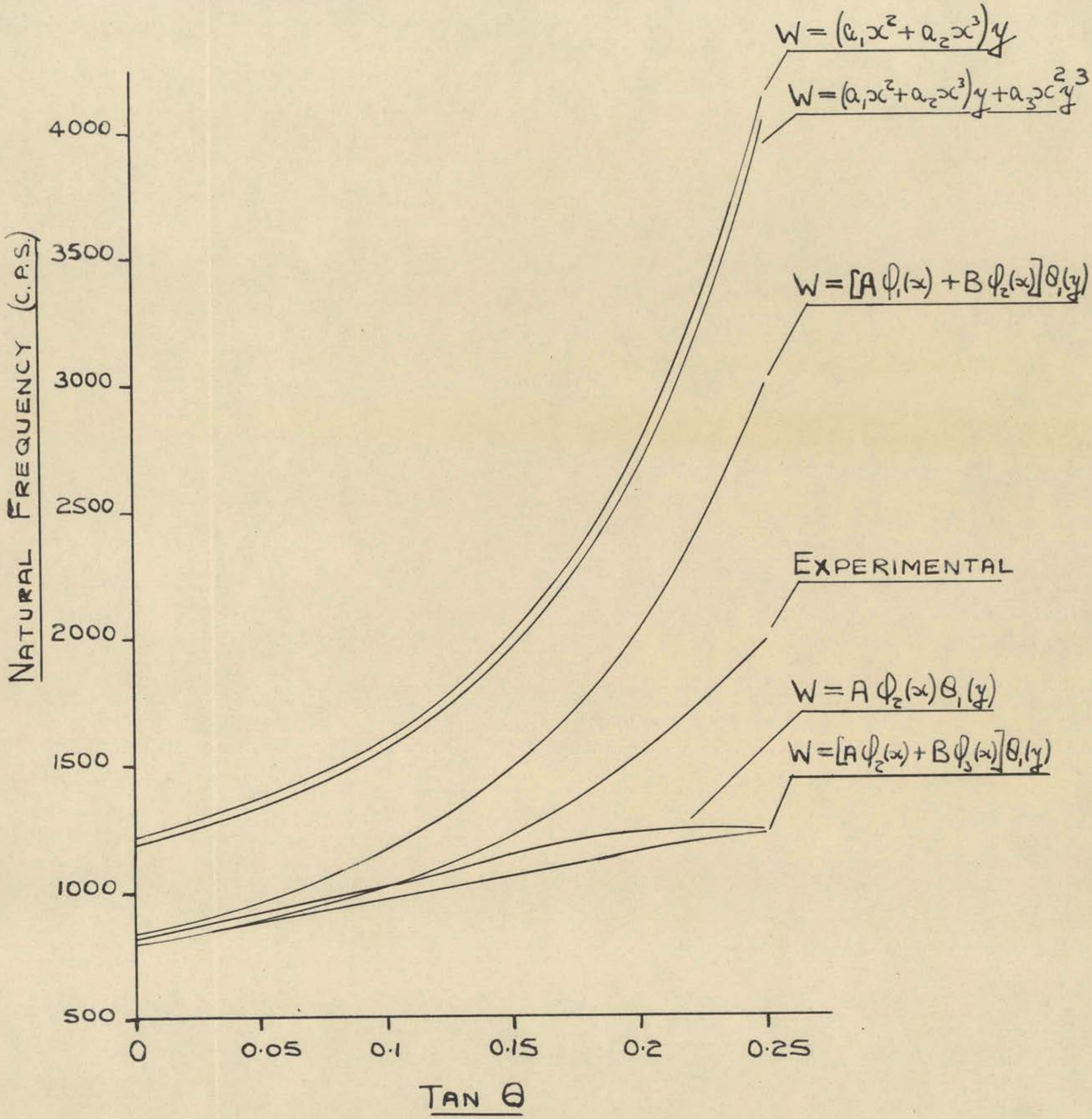
6.2 VARIATION OF NATURAL FREQUENCIES WITH ANGLE OF TAPER - COMPARISON OF EXPERIMENTAL AND CALCULATED RESULTS

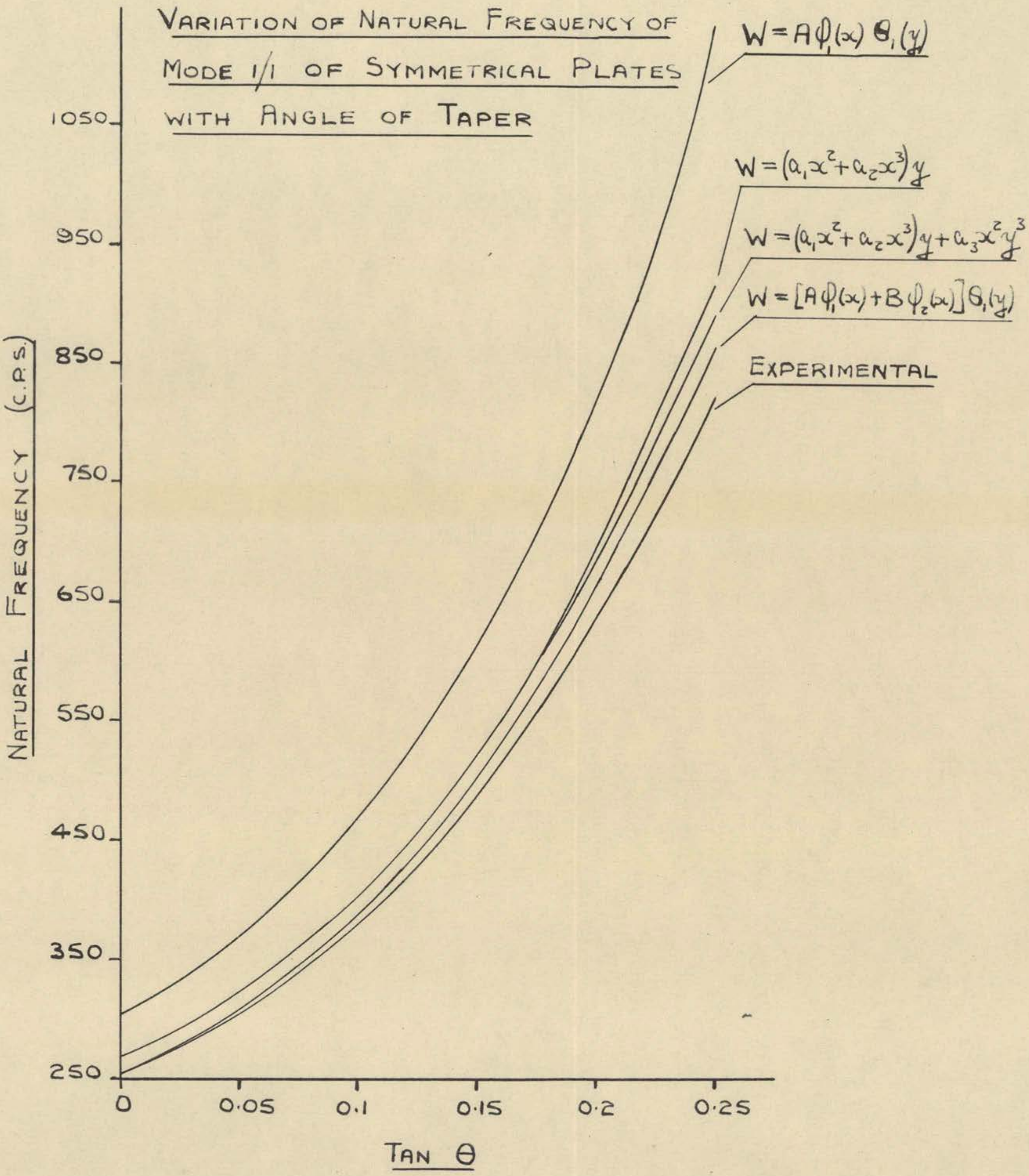
1) Modes m/0 of Symmetrical Plates

Tan θ	0	1/16	1/8	3/16	5/24	1/4
	Natural Frequency (c.p.s.)					
Mode 1/0						
Experimental	58	63.4	70.2	83.4	90.1	114
Calculated:-						
$W = A\phi_1(x)\theta_0(y)$	59.7	65.2	73.3	87.5	95.3	122
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	59.5	64.9	73	87	94.7	121
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	59.5					121
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	59.5					121
$W = a_1x^2 + a_2x^3$	59.6	64.9	73	87	94.7	121
$W = a_1x^2 + a_2x^3 + a_3x^4$	59.5	64.9	73	87	94.7	121
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	59.5					121
Mode 2/0						
Experimental	358	369	380	409	422	494
Calculated:-						
$W = A\phi_2(x)\theta_0(y)$	374	385	398	418	428	453
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	373	383	400	426	443	528
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	373					526
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	373					526
$W = a_1x^2 + a_2x^3$	590	583	573	565	566	630
$W = a_1x^2 + a_2x^3 + a_3x^4$	377	386	401	428	445	530
$W = a_1x^2 + a_3x^3 + a_3x^4 + a_4x^5$	376					528

Tan θ	0	1/16	1/8	3/16	5/24	1/4
	Natural Frequency (c.p.s.)					
Mode 3/0						
Experimental	1003	1011	1022	1046	1062	1193
Calculated:-						
$W = A\phi_3(x)\theta_0(y)$	1046	1056	1069	1087	1096	1115
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	1046					1283
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	1046					1280
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	1074					1315
Mode 4/0						
Experimental	1970	1966	1976	1996	2011	2202
Calculated:-						
$W = A\phi_4(x)\theta_0(y)$	2050	2059	2072	2090	2098	2118
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	2049					2371
Mode 5/0						
Experimental	3255	3237	3233		3250	3485
Calculated:-						
$W = A\phi_5(x)\theta_0(y)$	3387	3398	3411	3425	3438	3457
Mode 6/0						
Experimental	4823	4846	4757	4792	4807	5100
Calculated:-						
$W = A\phi_6(x)\theta_0(y)$	5062	5072	5084	5102	5110	5130
Mode 7/0						
Experimental	6690	6682	6668	6604	6632	6964
Calculated:- $W = A\phi_7(x)\theta_0(y)$	7070	7080	7092	7111	7118	7138

Variation of Natural Frequency of Mode 2/1 of Symmetrical Plates with Angle of Taper





VARIATION OF NATURAL FREQUENCY OF
MODE 1/1 OF SYMMETRICAL PLATES
WITH ANGLE OF TAPER
$W = A\phi_1(x)\,\theta_1(y)$
$W = (a_1x^2 + a_2x^3)y$
$W = (a_1x^2 + a_2x^3)y + a_3x^2y^3$
$W = [A\phi_1(x) + B\phi_2(x)]\,\theta_1(y)$
EXPERIMENTAL
NATURAL FREQUENCY (C.P.S.)
1050
950
850
750
650
550
450
350
250
0
0·05
0·1
0·15
0·2
0·25
TAN θ

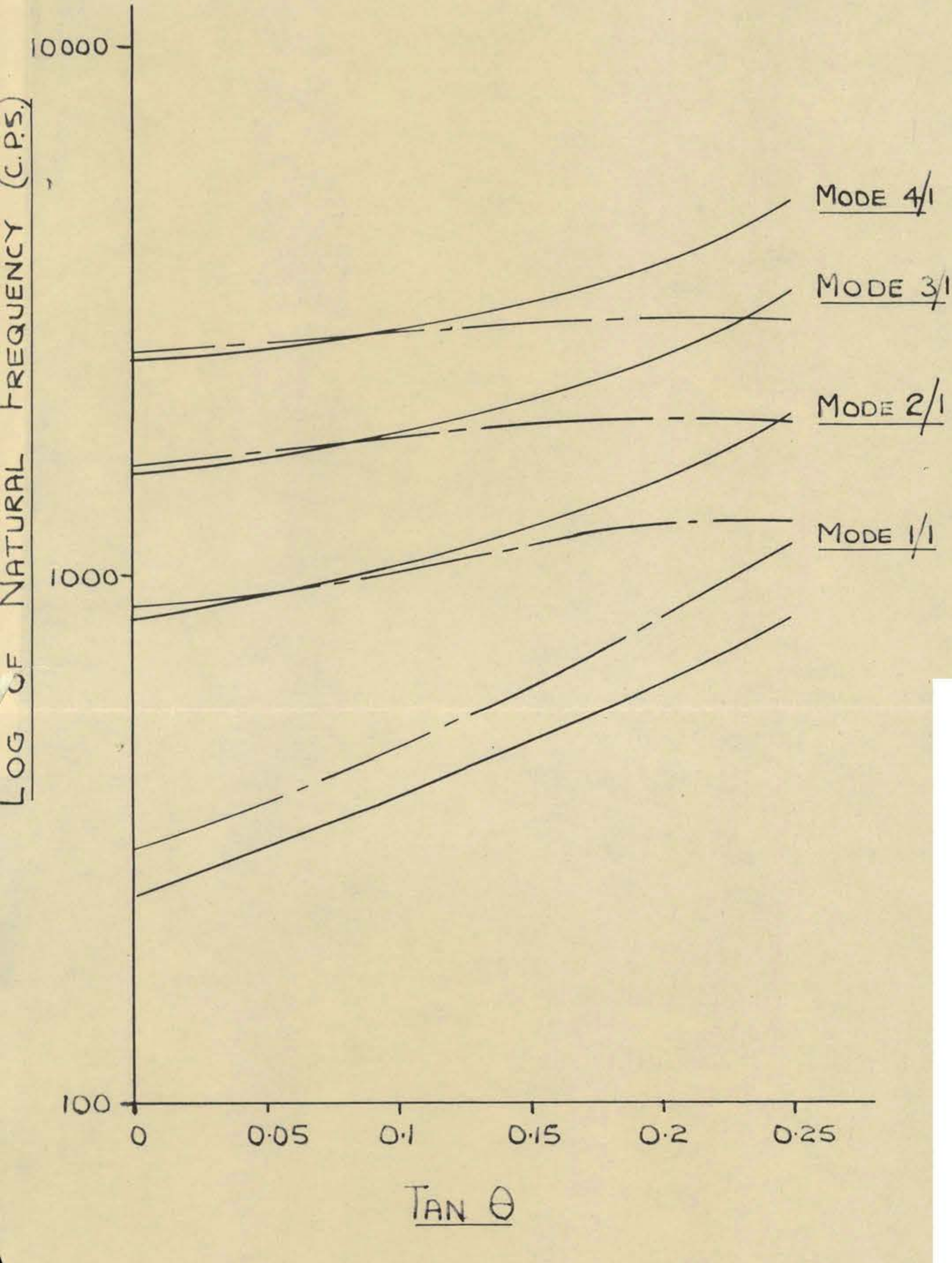
VARIATION OF NATURAL FREQUENCY OF MODES M/1 OF SYMMETRICAL PLATES WITH ANGLE OF TAPER

ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:-

$$W = A\psi_m(x)\,\theta_1(y)$$

——— EXPERIMENTAL FREQUENCIES

—·— CALCULATED FREQUENCIES

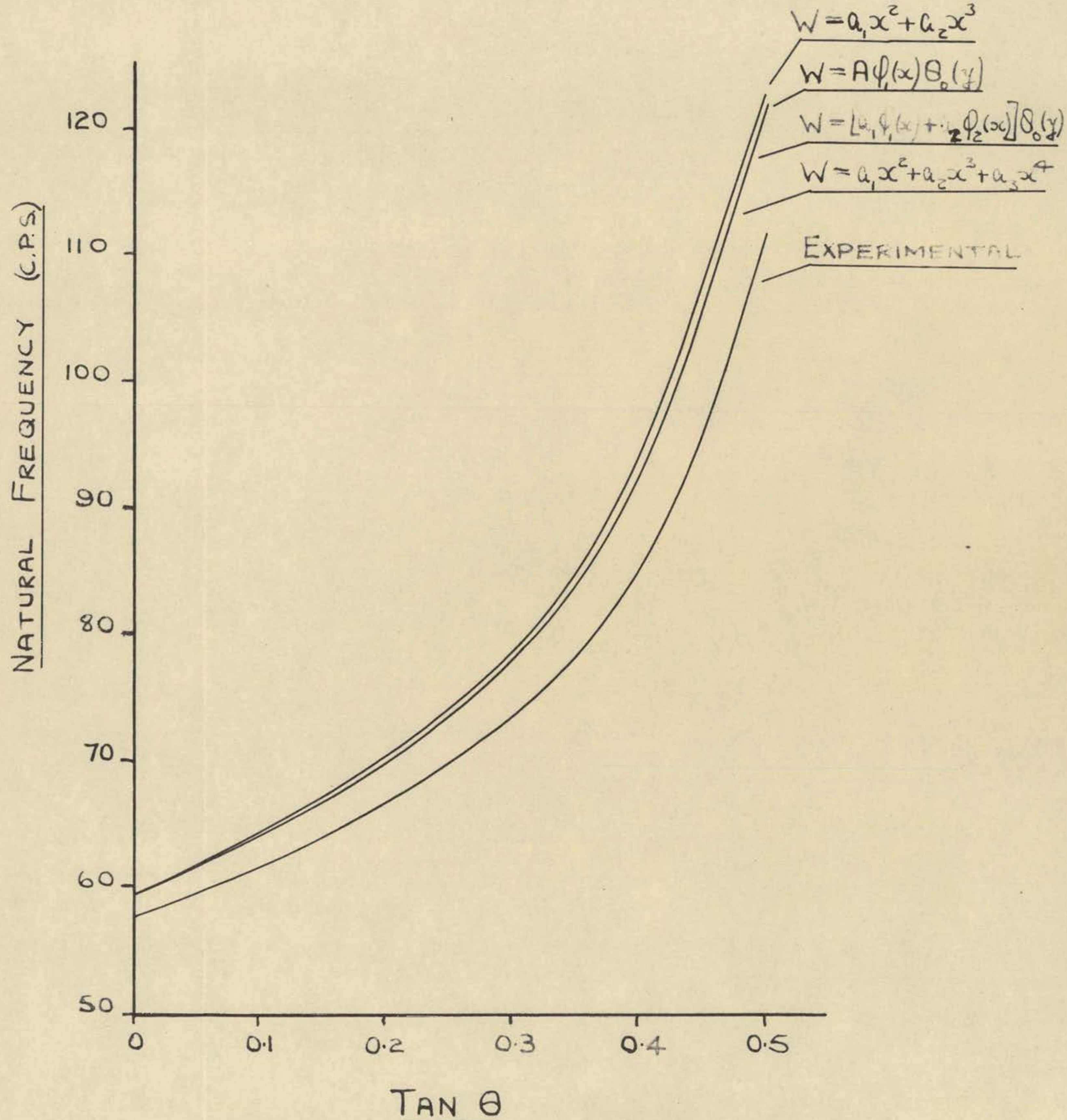


2) <u>Modes m/1 of Symmetrical Plates</u>

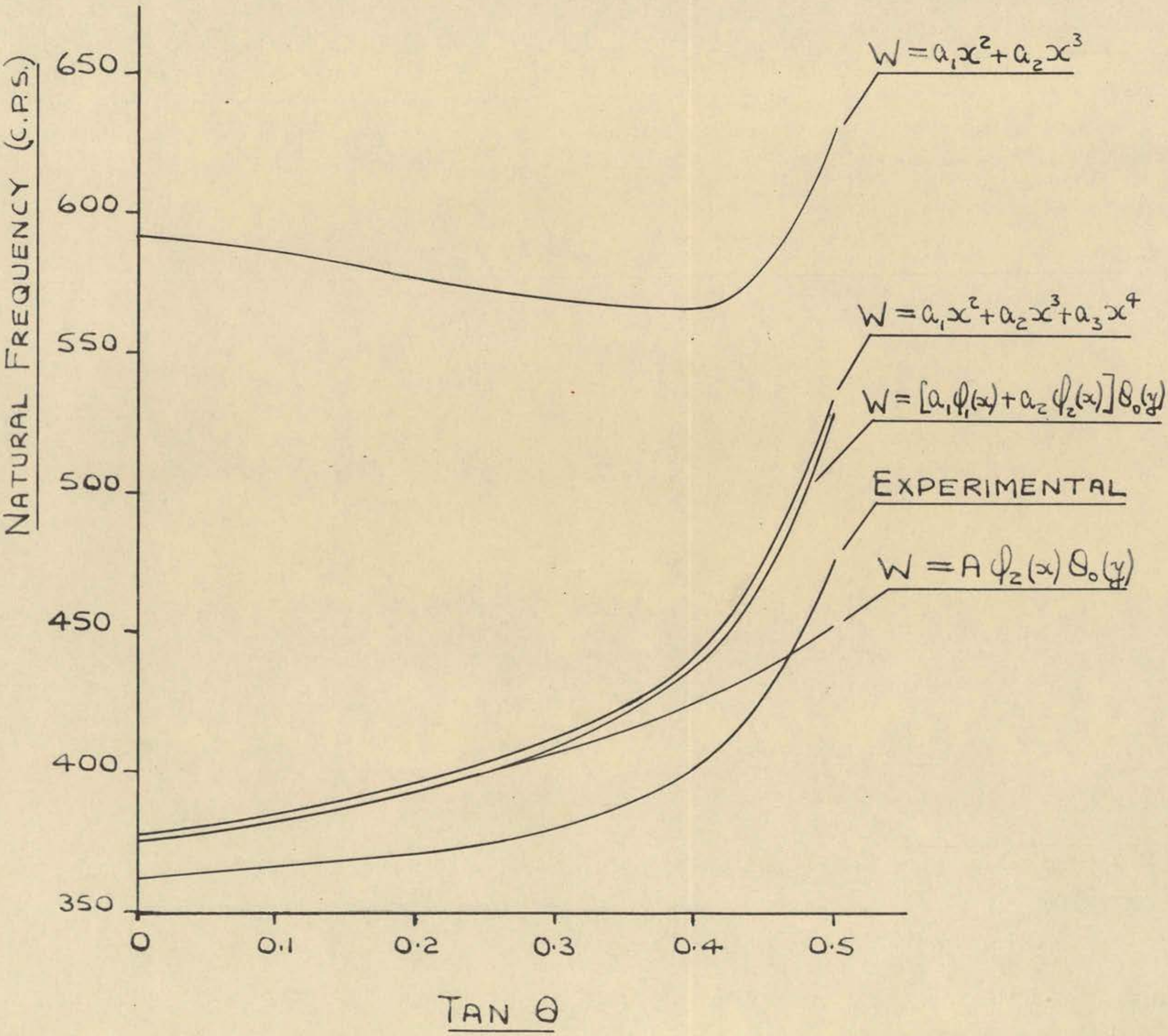
Tan θ	0	1/16	1/8	3/16	5/24	1/4
	Natural Frequency (c.p.s.)					
Mode 1/1						
Experimental	256	322	430	597	666	821
Calculated:-						
$W = A\phi_1(x)\theta_1(y)$	305	390	529	773	873	1138
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	254	324	436	615	692	865
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	254					861
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$	252					861
$W = [a_1x^2 + a_2x^3]y$	268	338	452	645		918
$W = [a_1x^2 + a_2x^3]y + a_3x^2y^3$	268	341	452	640		893
$W = [a_1x^2 + a_2x^3 + a_3x^4]y$	253					863
$W = [a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5]y$	257					859
Mode 2/1						
Experimental	820	939	1119	1446	1606	1994
Calculated:-						
$W = A\phi_2(x)\theta_1(y)$	874	989	1083	1228	1240	1253
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$	816	916	1026	1141	1177	1230
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	891	1049	1307	1851	2149	3003
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	832					2128
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$	831					2137
$W = [a_1x^2 + a_2x^3]y$	1227	1422	1761	2499		4130
$W = [a_1x^2 + a_2x^3]y + a_3x^2y^3$	1204	1406	1755	2423		4050
$W = [a_1x^2 + a_2x^3 + a_3x^4]y$	856					2432
$W = [a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5]y$	856					2142

Tan θ	0	1/16	1/8	3/16	5/24	1/4
	Natural Frequency (c.p.s.)					
Mode 3/1						
Experimental	1558	1728	1994	2473	2722	3416
Calculated:-						
$W = A\phi_3(x)\theta_1(y)$	1608	1739	1876	1967	1980	1967
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	1639					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$	1601					3679
$W = [a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5]y$	1842					4248
Mode 4/1						
Experimental	2545	2739	3079	3712	4056	5096
Calculated:-						
$W = A\phi_4(x)\theta_1(y)$	2636	2786	2938	3051	3070	3067
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$	2651					

Variation of Natural Frequency of Mode 1/0 of Right-Angled Plates with Angle of Taper



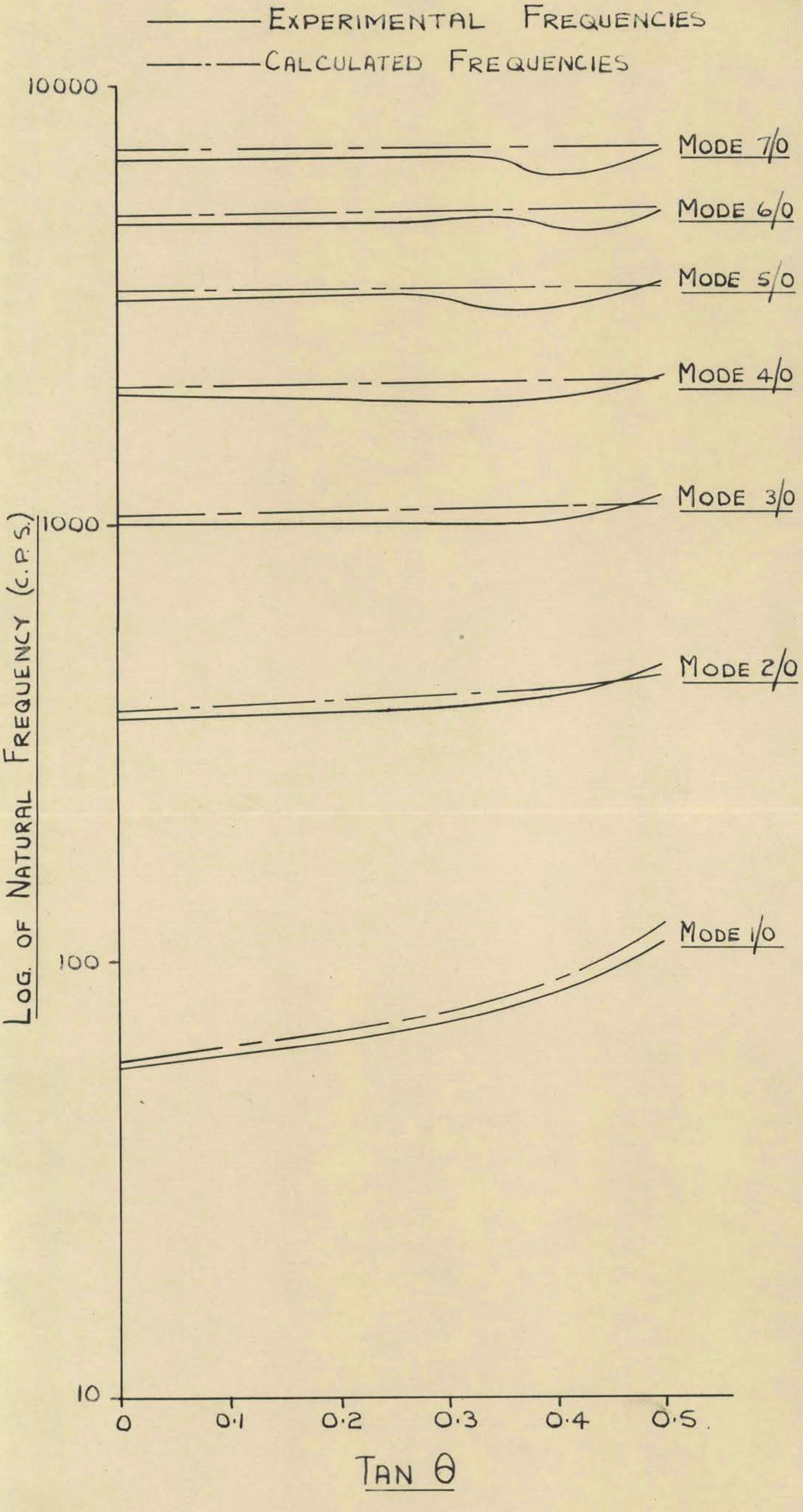
Variation of Natural Frequency of Mode 2/0 of Right-Angled Plates with Angle of Taper



VARIATION OF NATURAL FREQUENCY OF MODES M/O OF RIGHT-ANGLED PLATES WITH ANGLE OF TAPER

ASSUMED DEFLECTED FORM FOR CALCULATED FREQUENCIES:-

$$W = A\,\phi_m(x)\,\theta_0(y)$$

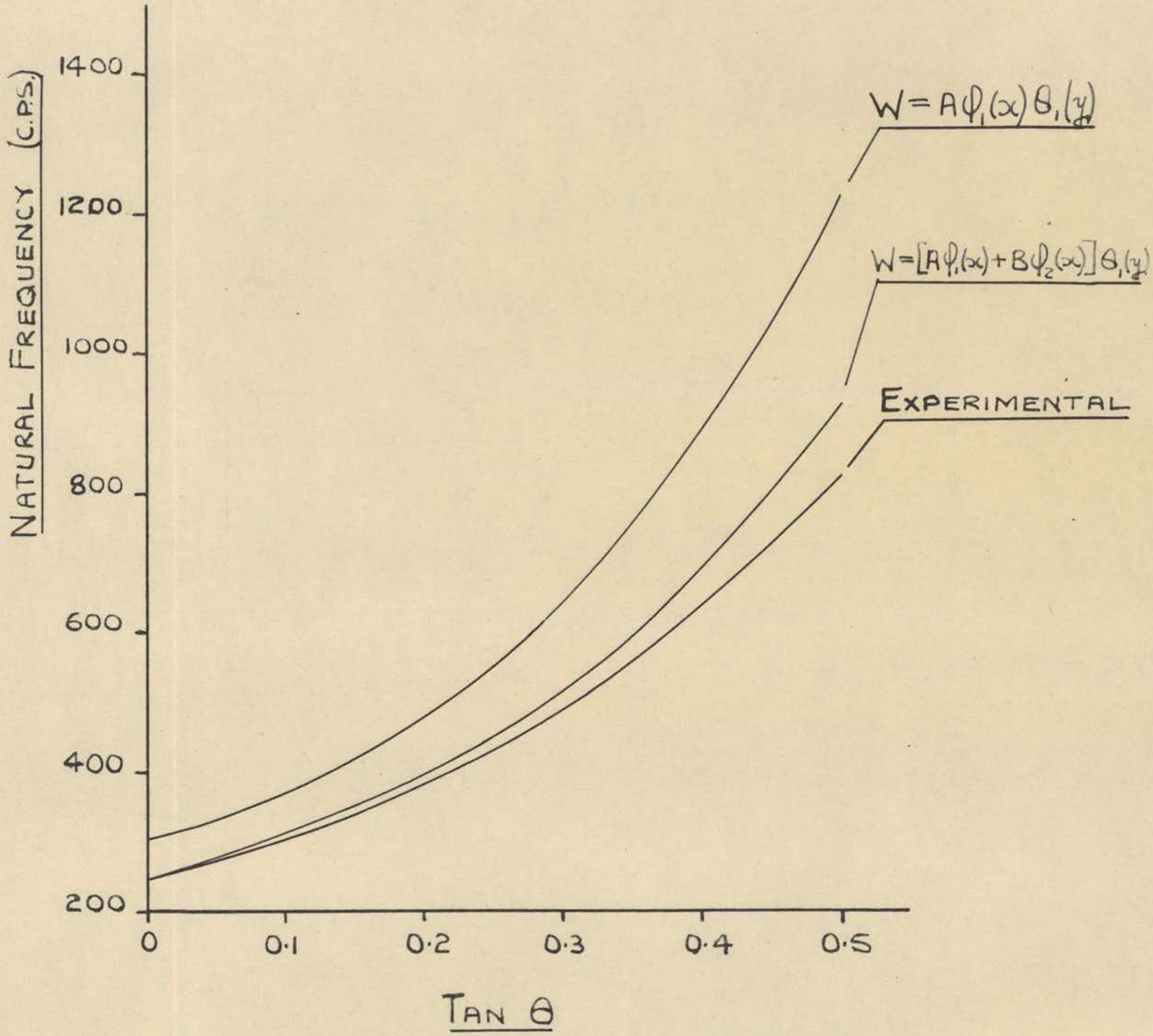


3) Modes m/O of Right-angled Plates

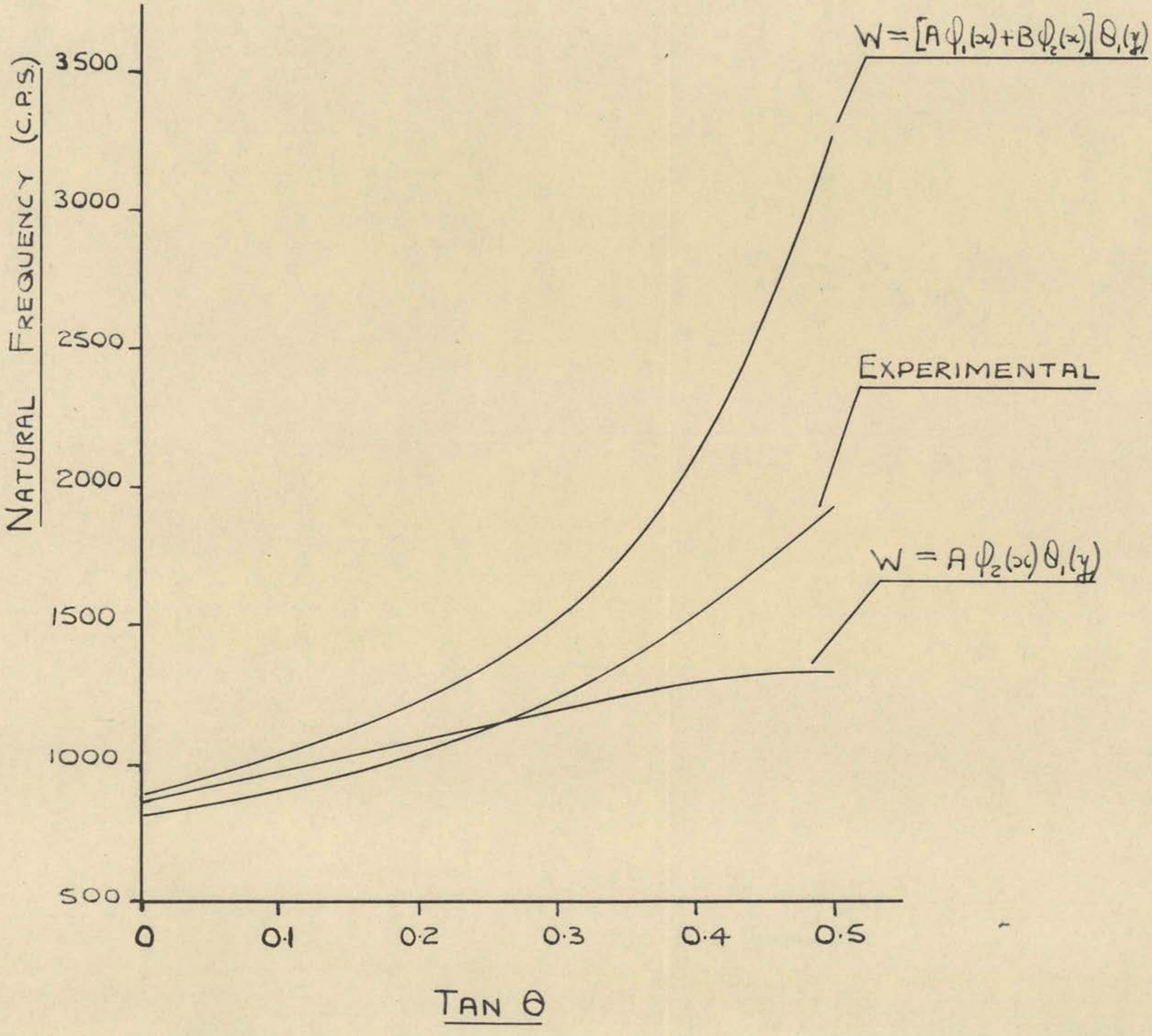
Tan θ	0	1/8	1.1/4	1/3	5/12	1/2
	Natural Frequency (c.p.s.)					
Mode 1/0						
Experimental	57.9	62.4	71.6	76.4	87.9	112
Calculated:-						
$W = A\phi_1(x)\theta_0(y)$	59.6	65.1	75.7	81.9	95.5	122
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	59.6	65.1		81.6	95.1	122
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	59.6					122
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	59.6					122
$W = a_1x^2 + a_2x^3$	59.6	65.4		82.2	95.1	123
$W = a_1x^2 + a_2x^3 + a_3x^4$	59.6	65.1		81.6	95.1	122
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	59.5					122
Mode 2/0						
Experimental	362	368	376	392	409	477
Calculated:-						
$W = A\phi_2(x)\theta_0(y)$	375	385	403	412	429	453
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	375	385		417	445	530
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	375					528
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	375					528
$W = a_1x^2 + a_2x^3$	592	585		569	568	632
$W = a_1x^2 + a_2x^3 + a_3x^4$	378	388		419	447	532
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	377					530

Tan θ	0	1/8	1.1/4	1/3	5/12	1/2
	Natural Frequency (c.p.s.)					
Mode 3/0						
Experimental	1006	1015	1000	1013	1032	1157
Calculated:-						
$W = A\phi_3(x)\theta_0(y)$	1050	1060	1076	1084	1099	1120
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	1050					1288
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	1050					1285
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	1078					1320
Mode 4/0						
Experimental	1969	1967	1905	1923	1959	2185
Calculated:-						
$W = A\phi_4(x)\theta_0(y)$	2058	2067	2084	2092	2107	2127
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	2058					2380
Mode 5/0						
Experimental	3251	3239	3327	3067	3164	3559
Calculated:- $W = A\phi_5(x)\theta_0(y)$	3402	3411	3428	3436	3451	3471
Mode 6/0						
Experimental	4817	4855	4857	5019	4660	5250
Calculated:- $W = A\phi_6(x)\theta_0(y)$	5082	5092	5108	5116	5131	5150
Mode 7/0						
Experimental	6695	6691		6801	6361	7220
Calculated:- $W = A\phi_7(x)\theta_0(y)$	7098	7108	7124	7132	7147	7166

VARIATION OF NATURAL FREQUENCY OF MODE 1/1 OF RIGHT-ANGLED PLATES WITH ANGLE OF TAPER



VARIATION OF NATURAL FREQUENCY OF MODE 2/1 OF RIGHT-ANGLED PLATES WITH ANGLE OF TAPER



6.3 COMPARISON OF EXPERIMENTAL AND CALCULATED FREQUENCIES OF RECTANGULAR, TRAPEZOIDAL AND TRIANGULAR PLATES

1) Symmetrical Plates

a) Rectangular Plate

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	58	358	1003	1970	3255	4823	6690
Calculated:-							
$W = A\phi_m(x)\theta_0(y)$	59.7	374	1046	2050	3387	5062	7070
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	59.5	373					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	59.5	373	1046				
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	59.5	373	1046	2049			
$W = a_1x^2 + a_2x^3$	59.6	590					
$W = a_1x^2 + a_2x^3 + a_3x^4$	59.5	377					
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	59.5	376	1074				

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	256	820	1558	2545	3808	5353	7251
Calculated:-							
$W = A\phi_m(x)\theta_1(y)$	305	874	1608	2636			
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	254	891					
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$		816					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	254	832	1639				
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$	252	831	1601	2651			
$W = [a_1x^2 + a_2x^3]y$	268	1227					
$W = [a_1x^2 + a_2x^3]y + a_3x^2y^3$	268	1204					
$W = [a_1x^2 + a_2x^3 + a_3x^4]y$	253	856					
$W = [a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5]y$	257	856	1842				

b) Symmetrical Trapezoidal Plate No. 1.

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	63.4	369	1011	1966	3237	4846	6682
Calculated:-							
$W = A\phi_m(x)\theta_0(y)$	65.2	385	1056	2059	3398	5072	7080
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	64.9	383					
$W = a_1x^2 + a_2x^3$	64.9	583					
$W = a_1x^2 + a_2x^3 + a_3x^4$	64.9	386					

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	322	939	1728	2739	4016	5577	7417
Calculated:-							
$W = A\phi_m(x)\theta_1(y)$	390	989	1739	2786			
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	324	1049					
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$		916					
$W = [a_1x^2 + a_2x^3]y$	338	1422					
$W = [a_1x^2 + a_2x^3]y + a_3x^2y^3$	341	1406					

c) Symmetrical Trapezoidal Plate No. 2

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	70.2	380	1022	1976	3233	4757	6668
Calculated:-							
$W = A\phi_m(x)\theta_0(y)$	73.3	398	1069	2072	3411	5084	7092
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	73	400					
$W = a_1x^2 + a_2x^3$	73	573					
$W = a_1x^2 + a_2x^3 + a_3x^4$	73	401					

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	430	1119	1994	3079	4395	5968	7776
Calculated:-							
$W = A\phi_m(x)\theta_1(y)$	529	1083	1876	2938			
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	436	1307					
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$		1026					
$W = [a_1x^2 + a_2x^3]y$	452	1761					
$W = [a_1x^2 + a_2x^3]y + a_3x^2y^3$	452	1755					

4) **Modes m/1 of Right-angled Plates**

Tan θ	0	1/8	1.1/4	1/3	5/12	1/2
	Natural Frequency (c.p.s.)					
Mode 1/1						
Experimental	255	319	461	535	668	820
Calculated:-						
$W = A\phi_1(x)\theta_1(y)$	306	394		712	942	1234
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	255	328		565		928
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	255					918
$W = [a_1x^2 + a_2x^3]y$	269					1048
$W = [a_1x^2 + a_2x^3 + a_3x^4]y$	255					926
Mode 2/1						
Experimental	824	930	1185	1323	1598	1924
Calculated:-						
$W = A\phi_2(x)\theta_1(y)$	877	1000		1239	1318	1343
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$	820					1312
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	894	1055		1671		3260
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	835					2275
$W = [a_1x^2 + a_2x^3]y$	1230					4737
$W = [a_1x^2 + a_2x^3 + a_3x^4]y$	859					2909
Mode 3/1						
Experimental	1558	1714	2112	2291	2684	3137
Calculated:- $W = A\phi_3(x)\theta_1(y)$	1614	1754		2000	2067	2081
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	1645					
Mode 4/1						
Experimental	2544	2724	3034	3515	4029	4670
Calculated:- $W = A\phi_4(x)\theta_1(y)$	2648	2807		3092	3176	3207

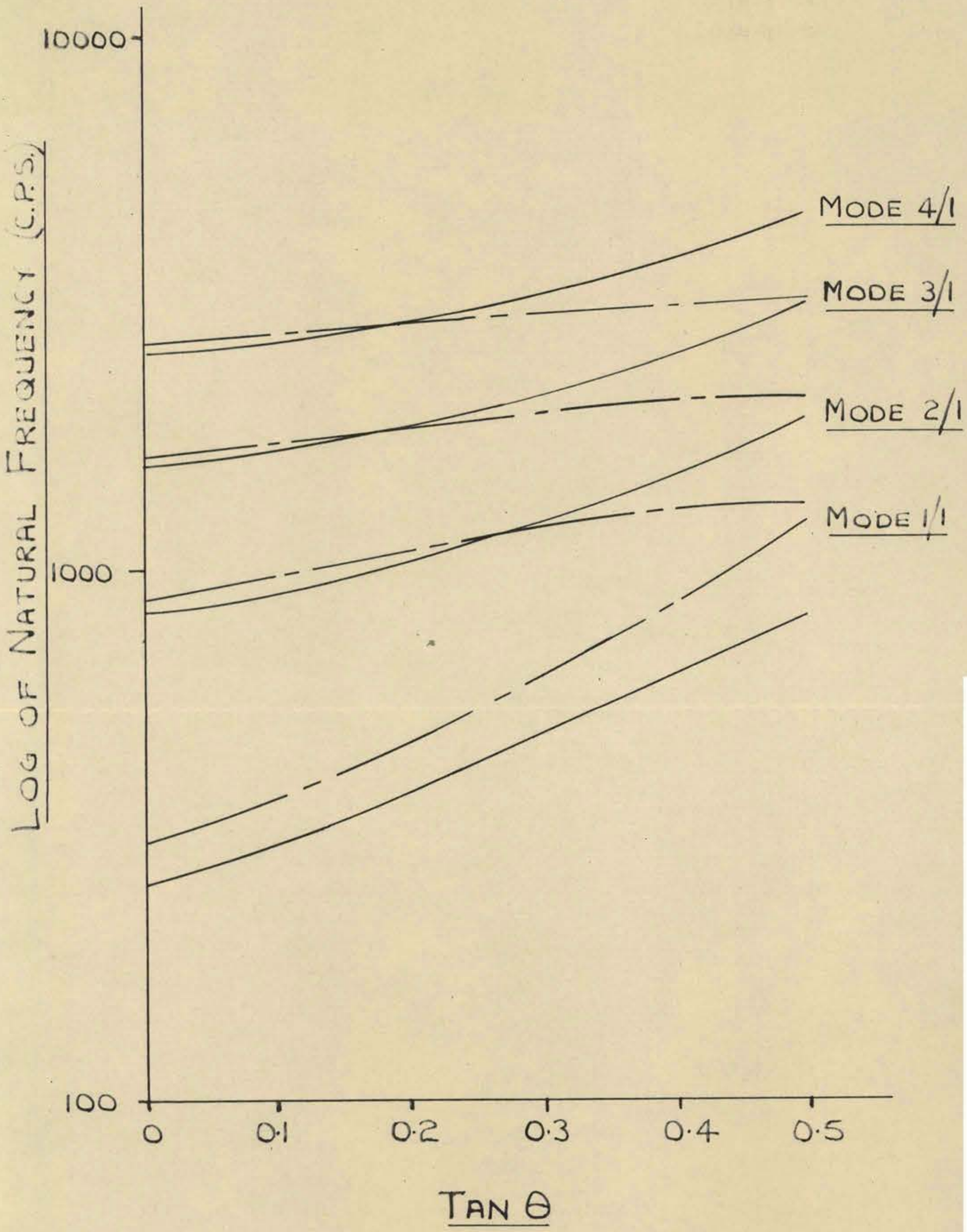
Variation of Natural Frequency of Modes M/1 of Right-Angled Plates with Angle of Taper

Assumed Deflected Form for Calculated Frequencies :—

$W = A\,\phi_m(x)\,\theta_1(y)$

——— Experimental Frequencies

—-— Calculated Frequencies



d) **Right-angled Trapezoidal Plate No. 3**

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	76.4	392	1013	1923	3067	5019	6801
Calculated:-							
$W = A\phi_m(x)\Theta_0(y)$	81.9	412	1084	2092	3436	5116	7132
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\Theta_0(y)$	81.6	417					
$W = a_1x^2 + a_2x^3$	82.2	569					
$W = a_1x^2 + a_2x^3 + a_3x^4$	81.6	419					

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	535	1323	2291	3515	4461	6002	8054
Calculated:-							
$W = A\phi_m(x)\Theta_1(y)$	712	1239	2000	3092			
$W = [A\phi_1(x) + B\phi_2(x)]\Theta_1(y)$	565	1671					

e) **Right-angled Trapezoidal Plate No. 4**

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	87.9	409	1032	1959	3164	4660	6361
Calculated:-							
$W = A\phi_m(x)\Theta_0(y)$	95.5	429	1099	2107	3451	5131	7147
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\Theta_0(y)$	95.1	445					
$W = a_1x^2 + a_2x^3$	95.1	568					
$W = a_1x^2 + a_2x^3 + a_3x^4$	95.1	447					

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	668	1598	2684	4029	5689		
Calculated:- $W = A\phi_m(x)\Theta_1(y)$	942	1318	2067	3176			

f) **Right-angled Triangular Plate**

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	112	477	1157	2185	3559	5250	7220
Calculated:-							
$W = A\phi_m(x)\Theta_0(y)$	122	453	1120	2127	3471	5150	7166
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\Theta_0(y)$	122	530					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\Theta_0(y)$	122	528	1288				
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\Theta_0(y)$	122	528	1285	2380			
$W = a_1x^2 + a_2x^3$	123	632					
$W = a_1x^2 + a_2x^3 + a_3x^4$	122	532					
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	122	530	1320				

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	820	1924	3137	4670			
Calculated:-							
$W = A\phi_m(x)\Theta_1(y)$	1234	1343	2081	3207			
$W = [A\phi_1(x) + B\phi_2(x)]\Theta_1(y)$	928	3260					
$W = [A\phi_2(x) + B\phi_3(x)]\Theta_1(y)$		1312					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\Theta_1(y)$	918	2275					
$W = [a_1x^2 + a_2x^3]y$	1048	4737					
$W = [a_1x^2 + a_2x^3 + a_3x^4]y$	926	2909					

d) Symmetrical Trapezoidal Plate No. 3

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	83.4	409	1046	1996		4792	6604
Calculated:-							
$W = A\phi_m(x)\theta_0(y)$	87.5	418	1087	2090	3425	5102	7111
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	87	426					
$W = [a_1x^2 + a_2x^3]$	87	565					
$W = [a_1x^2 + a_2x^3 + a_3x^4]$	87	428					

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	597	1446	2473	3712	5175	6833	
Calculated:-							
$W = A\phi_m(x)\theta_1(y)$	773	1228	1967	3051			
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	615	1851					
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$		1141					
$W = [a_1x^2 + a_2x^3]y$	645	2499					
$W = [a_1x^2 + a_2x^3]y + a_3x^2y^3$	640	2423					

e) Symmetrical Trapezoidal Plate No. 4

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	90.1	422	1062	2011	3250	4807	6632
Calculated:-							
$W = A\phi_m(x)\theta_0(y)$	95.3	428	1096	2098	3438	5110	7118
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	94.7	443					
$W = a_1x^2 + a_2x^3$	94.7	566					
$W = a_1x^2 + a_2x^3 + a_3x^4$	94.7	445					

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1
	Natural Frequency (c.p.s.)				
Experimental	666	1606	2722	4056	5607
Calculated:-					
$W = A\phi_m(x)\theta_1(y)$	873	1240	1980	3070	
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	692	2149			
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$		1177			

f) Isosceles Triangular Plate

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	114	494	1193	2202	3485	5100	6964
Calculated:-							
$W = A\phi_m(x)\theta_0(y)$	122	453	115	2118	3457	5130	7138
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	121	528					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	121	526	1283				
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	121	526	1280	2371			
$W = a_1x^2 + a_2x^3$	122	630					
$W = a_1x^2 + a_2x^3 + a_3x^4$	121	530					
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	121	528	1315				

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	821	1994	3416	5096	7009		
Calculated:-							
$W = A\phi_m(x)\theta_1(y)$	1138	1253	1967	3067			
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	865	3003					
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$		1230					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	861	2128					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$	861	2137	3679				
$W = [a_1x^2 + a_2x^3]y$	918	4130					
$W = [a_1x^2 + a_2x^3]y + a_3x^2y^3$	893	4050					
$W = [a_1x^2 + a_2x^3 + a_3x^4]y$	863	2432					
$W = [a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5]y$	859	2142	4248				

2) Right-angled Plates

a) Rectangular Plate

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	57.9	362	1006	1969	3251	4817	6695
Calculated:-							
$W = A\phi_m(x)\theta_0(y)$	59.6	375	1050	2058	3402	5082	7098
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	59.6	375					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$	59.6	375	1050				
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	59.6	375	1050	2058			
$W = a_1x^2 + a_2x^3$	59.6	592					
$W = a_1x^2 + a_2x^3 + a_3x^4$	59.6	378					
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	59.5	377	1078				

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	255	824	1558	2544	3803	5337	7235
Calculated:-							
$W = A\phi_m(x)\theta_1(y)$	306	877	1614	2648			
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	255	894					
$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$		820					
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$	255	835	1645				
$W = [a_1x^2 + a_2x^3]y$	269	1230					
$W = [a_1x^2 + a_2x^3 + a_3x^4]y$	255	859					

b) Right-angled Trapezoidal Plate No. 1

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	62.4	368	1015	1967	3239	4855	6691
Calculated:-							
$W = A\phi_m(x)\theta_0(y)$	65.1	385	1060	2067	3411	5092	7108
$W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$	65.1	385					
$W = a_1x^2 + a_2x^3$	65.4	585					
$W = a_1x^2 + a_2x^3 + a_3x^4$	65.1	388					

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	319	930	1714	2724	3988	5519	7325
Calculated:-							
$W = A\phi_m(x)\theta_1(y)$	394	1000	1754	2807			
$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$	328	1055					

c) Right-angled Trapezoidal Plate No. 2

Modes m/0

Mode	1/0	2/0	3/0	4/0	5/0	6/0	7/0
	Natural Frequency (c.p.s.)						
Experimental	71.6	376	1000	1905	3327	4857	
Calculated $W = A\phi_m(x)\theta_0(y)$	75.7	403	1076	2084	3428	5108	7124

Modes m/1

Mode	1/1	2/1	3/1	4/1	5/1	6/1	7/1
	Natural Frequency (c.p.s.)						
Experimental	461	1185	2112	3034	4381	5911	7422

6.4 VALUES OF λ AND CORRESPONDING VALUES OF COEFFICIENTS IN EXPRESSIONS FOR PLATE DEFLECTED FORM

6.4.1. Values of coefficients from series of beam functions

1) Symmetrical Plates

a) Mode 1/0

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	12.362	14.718	18.621	26.479	31.276	51.224
a_1	1.000	1.000	1.000	1.000	1.000	1.000
a_2	0	-0.008	-0.015	-0.018	-0.023	-0.020

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$

tan θ	0	1/4
λ	12.362	51.215
a_1	1.000	1.000
a_2	0	-0.014
a_3	0	-0.001

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$

tan θ	0	1/4
λ	12.362	51.215
a_1	1.000	1.000
a_2	0	-0.014
a_3	0	-0.001
a_4	0	0

b) Mode 2/0

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	485.518	513.186	556.541	640.358	693.055	971.598
a_1	0	-0.041	-0.113	-0.268	-0.361	-0.770
a_2	1.000	1.000	1.000	1.000	1.000	1.000

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$

tan θ	0	1/4
λ	485.518	964.373
a_1	0	-0.775
a_2	1.000	1.000
a_3	0	-0.029

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$

tan θ	0	1/4
λ	485.518	963.734
a_1	0	-0.775
a_2	1.000	1.000
a_3	0	-0.030
a_4	0	-0.001

c) Mode 3/0

Assumed deflected form:- $W=[a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)]\theta_0(y)$

tan θ	0	1/4
λ	3806.545	5727.968
a_1	0	0.652
a_2	0	-0.687
a_3	1.000	1.000

Assumed deflected form:- $W=[a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)+a_4\phi_4(x)]\theta_0(y)$

tan θ	0	1/4
λ	3806.545	5702.349
a_1	0	0.658
a_2	0	-0.693
a_3	1.000	1.000
a_4	0	-0.013

d) Mode 4/0

Assumed deflected form:- $W=[a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)+a_4\phi_4(x)]\theta_0(y)$

tan θ	0	1/4
λ	14617.269	19562.783
a_1	0	-0.571
a_2	0	0.577
a_3	0	-0.649
a_4	1.000	1.000

e) Mode 1/1

Assumed deflected form:- $W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	223.950	366.127	661.510	1314.708	1664.623	2603.977
A	1.000	1.000	1.000	1.000	1.000	1.000
B	0.203	0.212	0.225	0.274	0.323	0.351

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$

tan θ	0	1/4
λ	223.702	2583.358
a_1	1.000	1.000
a_2	0.209	0.338
a_3	0.001	0.002

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$

tan θ	0	1/4
λ	220.755	2580.607
a_1	1.000	1.000
a_2	0.203	0.320
a_3	-0.007	0.005
a_4	0.003	-0.009

f) Mode 2/1

Assumed deflected form:- $W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	2764.639	3830.381	5951.552	11739.698	16072.565	31395.924
A	-0.203	-0.384	-0.737	-1.393	-1.768	-2.281
B	1.000	1.000	1.000	1.000	1.000	1.000

Assumed deflected form:- $W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	2318.420	2918.656	3663.363	4533.717	4821.473	5265.572
A	1.000	1.000	1.000	1.000	1.000	1.000
B	0.230	0.254	0.290	0.310	0.245	0.204

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$

tan θ	0	1/4
λ	2411.961	15757.856
a_1	-0.207	-6.248
a_2	1.000	1.000
a_3	0.237	2.337

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$

tan θ	0	1/4
λ	2403.614	15894.392
a_1	-0.204	-6.168
a_2	1.000	1.000
a_3	0.226	2.312
a_4	-0.028	0.064

g) Mode 3/1

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$

tan θ	0
λ	9354.399
a_1	-0.041
a_2	-0.211
a_3	1.000

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$

tan θ	0	1/4
λ	8927.933	47092.772
a_1	0.034	-7.844
a_2	-0.217	3.304
a_3	1.000	1.000
a_4	0.181	-3.047

h) Mode 4/1

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$

tan θ	0
λ	24660.169
a_1	-0.020
a_2	0.052
a_3	-0.158
a_4	1.000

2) Right-angled Plates

a) Mode 1/0

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x)]\,\theta_0(y)$

tan θ	0	1/8	1/3	5/12	1/2
λ	12.362	14.718	23.094	31.276	51.224
a_1	1.000	1.000	1.000	1.000	1.000
a_2	0	-0.008	-0.016	-0.023	-0.020

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\,\theta_0(y)$

tan θ	0	1/2
λ	12.362	51.215
a_1	1.000	1.000
a_2	0	-0.014
a_3	0	-0.001

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\,\theta_0(y)$

tan θ	0	1/2
λ	12.362	51.215
a_1	1.000	1.000
a_2	0	-0.014
a_3	0	-0.001
a_4	0	0

b) Mode 2/0

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x)]\theta_0(y)$

tan θ	0	1/8	1/3	5/12	1/2
λ	485.518	513.186	604.367	693.055	971.598
a_1	0	-0.041	-0.200	-0.361	-0.770
a_2	1.000	1.000	1.000	1.000	1.000

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$

tan θ	0	1/2
λ	485.518	964.373
a_1	0	-0.775
a_2	1.000	1.000
a_3	0	-0.029

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$

tan θ	0	1/2
λ	485.518	963.734
a_1	0	-0.775
a_2	1.000	1.000
a_3	0	-0.030
a_4	0	-0.001

c) Mode 3/0

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_0(y)$

tan θ	0	1/2
λ	3806.545	5727.968
a_1	0	0.652
a_2	0	-0.687
a_3	1.000	1.000

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$

tan θ	0	1/2
λ	3806.545	5702.349
a_1	0	0.658
a_2	0	-0.693
a_3	1.000	1.000
a_4	0	-0.013

d) Mode 4/0

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$

tan θ	0	1/2
λ	14617.269	19562.783
a_1	0	-0.571
a_2	0	0.577
a_3	0	-0.649
a_4	1.000	1.000

e) Mode 1/1

Assumed deflected form:- $W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$

tan θ	0	1/8	1/3	1/2
λ	223.950	372.255	1169.656	3356.975
A	1.000	1.000	1.000	1.000
B	0.203	0.211	0.253	0.360

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$

tan θ	0	1/2
λ	223.702	3294.214
a_1	1.000	1.000
a_2	0.209	0.339
a_3	0.001	0.042

f) Mode 2/1

Assumed deflected form:- $W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$

tan θ	0	1/8	1/3	1/2
λ	2764.639	3895.161	10191.501	41396.234
A	-0.203	-0.391	-1.124	-2.293
B	1.000	1.000	1.000	1.000

Assumed deflected form:- $W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$

tan θ	0	1/2
λ	2318.420	6709.015
A	1.000	1.000
B	0.230	0.147

Assumed deflected form:- $W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)]\theta_1(y)$

tan θ	0	1/2
λ	2411.961	20179.208
a_1	-0.207	-6.656
a_2	1.000	1.000
a_3	0.237	2.519

6.4.2. Values of coefficients in series of algebraic functions

1) Symmetrical Plates

a) Mode 1/0

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	12.480	14.786	18.652	26.498	31.302	51.251
a_1	1.000	1.000	1.000	1.000	1.000	1.000
a_2	-0.383	-0.376	-0.368	-0.360	-0.356	-0.374

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	12.370	14.731	18.639	26.499	31.295	51.246
a_1	1.000	1.000	1.000	1.000	1.000	1.000
a_2	-0.543	-0.524	-0.436	-0.342	-0.300	-0.333
a_3	0.098	0.080	0.032	0.004	-0.004	-0.029

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4 + a_4\left(\frac{x}{l}\right)^5$

tan θ	0	1/4
λ	12.362	51.216
a_1	1.000	1.000
a_2	-0.432	-0.108
a_3	-0.064	-0.368
a_4	0.064	0.128

b) Mode 2/0

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	1211.752	1180.849	1143.666	1112.288	1117.841	1376.749
a_1	-0.821	-0.815	-0.804	-0.781	-0.766	-0.701
a_2	1.000	1.000	1.000	1.000	1.000	1.000

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4$

tan θ	0	1/16	1/8	3/16	5/24	1/4
λ	494.316	519.751	559.997	639.548	690.721	977.021
a_1	-0.502	-0.515	-0.514	-0.508	-0.503	-0.471
a_2	1.000	1.000	1.000	1.000	1.000	1.000
a_3	-0.459	-0.432	-0.432	-0.428	-0.428	-0.444

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4 + a_4\left(\frac{x}{l}\right)^5$

tan θ	0	1/4
λ	490.952	971.021
a_1	-0.405	-0.600
a_2	1.000	1.000
a_3	-0.780	0.056
a_4	0.208	-0.320

c) Mode 3/0

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4 + a_4\left(\frac{x}{l}\right)^5$

tan θ	0	1/4
λ	4013.505	6020.075
a_1	0.212	0.180
a_2	-0.827	-0.772
a_3	1.000	1.000
a_4	-0.380	-0.392

d) Mode 1/1

Assumed deflected form:- $W = \left[a_1\left(\frac{x}{\ell}\right)^2 + a_2\left(\frac{x}{\ell}\right)^3\right]\frac{2y}{b}$

tanθ	0	1/16	1/8	3/16	1/4
λ	249.385	402.891	713.801	1450.111	2934.625
a_1	1.000	1.000	1.000	1.000	1.000
a_2	-0.670	-0.684	-0.716	-0.756	-0.865

Assumed deflected form:- $W = \left[a_1\left(\frac{x}{\ell}\right)^2 + a_2\left(\frac{x}{\ell}\right)^3 + a_3\left(\frac{x}{\ell}\right)^4\right]\frac{2y}{b}$

tan θ	0	1/4
λ	222.761	2589.673
a_1	1.000	1.000
a_2	-1.339	-1.655
a_3	0.528	0.767

Assumed deflected form:- $W = \left[a_1\left(\frac{x}{\ell}\right)^2 + a_2\left(\frac{x}{\ell}\right)^3 + a_3\left(\frac{x}{\ell}\right)^4 + a_4\left(\frac{x}{\ell}\right)^5\right]\frac{2y}{b}$

tan θ	0	1/4
λ	229.973	2567.373
a_1	1.000	1.000
a_2	11.945	-2.358
a_3	-22.245	2.275
a_4	10.438	-0.836

e) Mode 2/1

Assumed deflected form:- $W = \left[a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3\right]\frac{2y}{b}$

tan θ	0	1/16	1/8	3/16	1/4
λ	5248.522	6489.261	10792.686	21727.544	59024.761
a_1	-0.803	-0.778	-0.731	-0.644	-0.514
a_2	1.000	1.000	1.000	1.000	1.000

Assumed deflected form:- $W = \left[a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4\right]\frac{2y}{b}$

tan θ	0	1/4
λ	2548.483	20546.505
a_1	-0.465	-0.332
a_2	1.000	1.000
a_3	-0.668	-0.782

Assumed deflected form:- $W = \left[a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4 + a_4\left(\frac{x}{l}\right)^5\right]\frac{2y}{b}$

tan θ	0	1/4
λ	2547.608	15965.221
a_1	-0.441	-0.235
a_2	1.000	1.000
a_3	-0.590	-1.264
a_4	0.062	0.512

2) **Right-angled Plates**

a) **Mode 1/0**

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3$

tan θ	0	1/8	1/3	5/12	1/2
λ	12.480	14.786	23.114	31.302	51.251
a_1	1.000	1.000	1.000	1.000	1.000
a_2	-0.383	-0.376	-0.362	-0.356	-0.374

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4$

tan θ	0	1/8	1/3	5/12	1/2
λ	12.370	14.731	23.114	31.295	51.246
a_1	1.000	1.000	1.000	1.000	1.000
a_2	-0.543	-0.524	-0.352	-0.300	-0.333
a_3	0.098	0.080	-0.008	-0.004	-0.029

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4 + a_4\left(\frac{x}{l}\right)^5$

tan θ	0	1/2
λ	12.362	51.216
a_1	1.000	1.000
a_2	-0.432	-0.108
a_3	-0.064	-0.368
a_4	0.064	0.128

b) Mode 2/0

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3$

tan θ	0	1/8	1/3	5/12	1/2
λ	1211.366	1180.849	1119.261	1117.841	1376.749
a_1	-0.821	-0.815	-0.791	-0.766	-0.701
a_2	1.000	1.000	1.000	1.000	1.000

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4$

tan θ	0	1/8	1/3	5/12	1/2
λ	494.316	519.751	604.999	690.721	977.021
a_1	-0.502	-0.515	-0.512	-0.503	-0.471
a_2	1.000	1.000	1.000	1.000	1.000
a_3	-0.459	-0.432	-0.426	-0.428	-0.444

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4 + a_4\left(\frac{x}{l}\right)^5$

tan θ	0	1/2
λ	490.952	971.021
a_1	-0.405	-0.600
a_2	1.000	1.000
a_3	-0.780	0.056
a_4	0.208	-0.320

c) Mode 3/0

Assumed deflected form:- $W = a_1\left(\frac{x}{l}\right)^2 + a_2\left(\frac{x}{l}\right)^3 + a_3\left(\frac{x}{l}\right)^4 + a_4\left(\frac{x}{l}\right)^5$

tan θ	0	1/2
λ	4013.505	6020.075
a_1	0.212	0.180
a_2	-0.827	-0.772
a_3	1.000	1.000
a_4	-0.380	-0.392

d) Mode 1/1

Assumed deflected form:- $W=\left[a_1\left(\frac{x}{l}\right)^2+a_2\left(\frac{x}{l}\right)^3\right]\frac{2y}{b}$

tan θ	0	1/2
λ	249.385	3820.732
a_1	1.000	1.000
a_2	-0.670	-0.867

Assumed deflected form:- $W=\left[a_1\left(\frac{x}{l}\right)^2+a_2\left(\frac{x}{l}\right)^3+a_3\left(\frac{x}{l}\right)^4\right]\frac{2y}{b}$

Tan θ	0	½
λ	222.761	3344.967
a_1	1.000	1.000
a_2	-1.339	-1.671
a_3	0.528	0.807

e) Mode 2/1

Assumed deflected form:- $W=\left[a_1\left(\frac{x}{l}\right)^2+a_2\left(\frac{x}{l}\right)^3\right]\frac{2y}{b}$

tan θ	0	½
λ	5248.522	78069.725
a_1	-0.803	-0.513
a_2	1.000	1.000

Assumed deflected form:- $W=\left[a_1\left(\frac{x}{l}\right)^2+a_2\left(\frac{x}{l}\right)^3+a_3\left(\frac{x}{l}\right)^4\right]\frac{2y}{b}$

tan θ	0	½
λ	2548.483	28745.021
a_1	-0.465	-0.322
a_2	1.000	1.000
a_3	-0.668	-0.648

CHAPTER VII

DISCUSSION OF RESULTS AND CONCLUSIONS

7.1. Comments on Experimental and Calculated Results

7.1.1. Single Term Approximation of the Deflected Form of the Vibrating Plate

Warburton (1954) used the Rayleigh-Ritz method to calculate the natural frequencies of all modes of vibration of rectangular plates having any combination of boundary conditions. He assumed that the wave forms of beams and rectangular plates are similar and used a combination of the appropriate beam functions to represent the deflected form of the vibrating plate. In the present research programme these deflected forms were used to calculate several of the natural frequencies of the families with no nodal lines and one nodal line perpendicular to the fixed edge of cantilevered, rectangular, trapezoidal and triangular plates. As a basis of comparison the natural frequencies of the plates under consideration were also obtained experimentally and on examination of these results (tables 1 to 8) the main points which may be noted are as follows:-

1) Mode 1/0 is the only mode of the family m/0 for which satisfactory results were obtained for all plate shapes. In the case of the symmetrical plates the discrepancy between the calculated and experimental results varies from 2.9% for the rectangular plate to 7% for the triangular plate (table 1, p. 32). The corresponding values for the right-angled plates are 2.9% and 8.9% (table 5, p. 39).

2) The experimental results show that the natural frequencies of the higher modes of the family m/0 (i.e. modes 2/0, 3/0, etc.) increase slightly, or tend to be constant, as the angle of taper is increased initially and then increase fairly rapidly as the shape of the plate approaches a triangle. This rapid increase in frequency was not predicted when the deflected forms which are being considered were

used to calculate the natural frequencies of these modes.

3) For all plate shapes, the value of the natural frequency of mode 1/1 which was calculated with the deflected form of the vibrating plate assumed to be

$$W = A\varphi_1(x)\,\theta_1(y)$$

(equations 4.5 and 4.16, pps. 33 and 41) is much higher than the experimental value. The discrepancy between the calculated and experimental values varies from 19.1% for the rectangular plate to 38.6% for the isosceles triangular plate (table 2, p. 34). The corresponding discrepancies for the right-angled plates are 20% and 50.4% (table 6, p. 42).

4) When the single term expression was used to calculate the natural frequencies of the higher modes of the family m/1, it was found that good agreement between calculated and experimental results was obtained only for the rectangular plate. For example, the calculated frequency of modes 2/1, 3/1 and 4/1 of the rectangular plate is respectively 6.6%, 3.2% and 3.6% higher than the experimental frequency. When the natural frequencies of these modes of the trapezoidal and triangular shaped plates were calculated, it was found that, in general, the calculated frequency was lower than the experimental frequency, and the discrepancy between the calculated and experimental frequencies increases as the angle of taper of the plate is increased. The experimental frequency of the isosceles triangular plate of modes 2/1, 3/1 and 4/1 is 37.2%, 42.4% and 39.8% higher than the calculated frequency (table 2, p. 34). The corresponding differences between the calculated and experimental frequencies of these modes of the right-angled triangular plate are 30.2%, 33.7% and 31.3% (table 6, p. 42).

5) The natural frequencies of modes 1/1 and 2/1 were calculated assuming

the deflected form of the vibrating plate to be

$$W = [A\phi_1(x) + B\phi_2(x)]\theta_1(y)$$

(equations 4.7 and 4.17, pps. 36 and 43). It was found that, for mode 1/1 of the symmetrical plates, the difference between calculated and experimental results varies from - 0.8% for the rectangular plate to 5.4% for the triangular plate (table 3, p. 36). For the right-angled plate these discrepancies are 0% and 13.2% (table 7, p. 43). For mode 2/1 the discrepancy between calculated and experimental results of the symmetrical plates is 8.7% for the rectangular plate and 50.6% for the triangular plate (table 3, p. 36). These discrepancies are 8.5% and 69.4% for the right-angled plates (table 7, p. 43).

6) The natural frequency of mode 2/1 was also calculated assuming the deflected form of the vibrating plate to be

$$W = [A\phi_2(x) + B\phi_3(x)]\theta_1(y)$$

(equations 4.8 and 4.18, pps. 35 and 43) and it was found that the values which were obtained were lower than the experimental values. The difference between the calculated and experimental frequencies increases as the angle of taper of the plate is increased. For the rectangular plates the difference between the calculated and experimental values is 0.5% (tables 4 and 8, pps. 36 and 44). The difference between the calculated and experimental results of the isosceles and right-angled triangular plates is 38.3% and 31.8% respectively (tables 4 and 8, pps. 36 and 44).

7.1.2. Series Approximation of the Deflected Form of the Vibrating Plate

In a further series of investigations a series of beam functions of the form

$$W = [a_1\phi_1(x) + a_2\phi_2(x) + \text{---}]\theta_\alpha(y)$$

and a series of algebraic functions of the form

$$W = a_1 x^2 + a_2 x^3 + - - -$$

were used to calculate the natural frequencies of modes m/0. A series of beam functions of the form

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + - - -]\, \theta_1(y)$$

and a series of algebraic functions of the form

$$W = (a_1 x^2 + a_2 x^3 + - - - - -)\, y$$

were also used to calculate the natural frequencies of the family m/1. Natural frequencies of the appropriate families of the plates under consideration were obtained using two, three and four terms of the appropriate series. On examination of the results which were obtained, the following points may be noted:-

1) The natural frequency of several of the modes were obtained from each deflected form and it was found that in each case the frequencies which were obtained for any particular mode tended to one value. When the results were compared with the experimental results it was found that the value, to which each one tended, was in good agreement with the corresponding experimental value.
2) From each particular deflected form there are obtained for each plate shape a number of frequencies of the particular family under consideration which are not improved to any appreciable extent by using a series with more terms to calculate the frequencies.
3) The calculated values of symmetrical plates of the family m/0 are, in general, the same as the calculated values of right-angled plates of the same area. The calculated values of modes m/1 of the symmetrical plates are lower than the calculated values of the corresponding modes of the right-angled plates of the same area.
4) With the exception of mode 1/1 of the rectangular plate, the calculated frequency of each mode for each plate is higher than the experimental value.

5) Although the difference between the calculated and experimental results increases as the angle of taper is increased, the calculated results in general agree closely with the experimental results.

6) The difference between the calculated and experimental results, which are obtained from each deflected form, increases slightly for the higher modes of the family.

7) The experimental frequencies of the right-angled plates tend to be slightly lower than the corresponding frequencies of the symmetrical plates of the same area. Where there is any noticable difference it is about 2% or 3% and in general the difference in the experimental frequencies of the right-angled and symmetrical plates increases as the angle of taper is increased.

Examination of the results has shown that the single term deflected forms, which have proved successful for the calculation of the natural frequencies of rectangular plates, do not, in general, give satisfactory results when they are used to calculate the natural frequencies of families m/0 and m/1 of triangular and trapezoidal plates. The only modes for which satisfactory results were obtained for all plate shapes were modes 1/0 and 1/1. Satisfactory results were, however, obtained for the higher modes of the families m/0 and m/1 if a series of beam functions or a series of algebraic functions is used to calculate the natural frequencies.

7.2. Discussion of Effects of Assumed Deflected Forms and Nodal Patterns on Calculated Natural Frequencies.

The use of the single term deflected form to calculate the natural frequencies of cantilevered tapered plates assumes that the deflected forms of cantilevered rectangular plates and cantilevered tapered plates are similar to those of cantilevered beams, and is equivalent to assuming that

FIGURE 7.4 VARIATION OF NODAL PATTERN OF MODE 3/1 WITH ANGLE OF TAPER

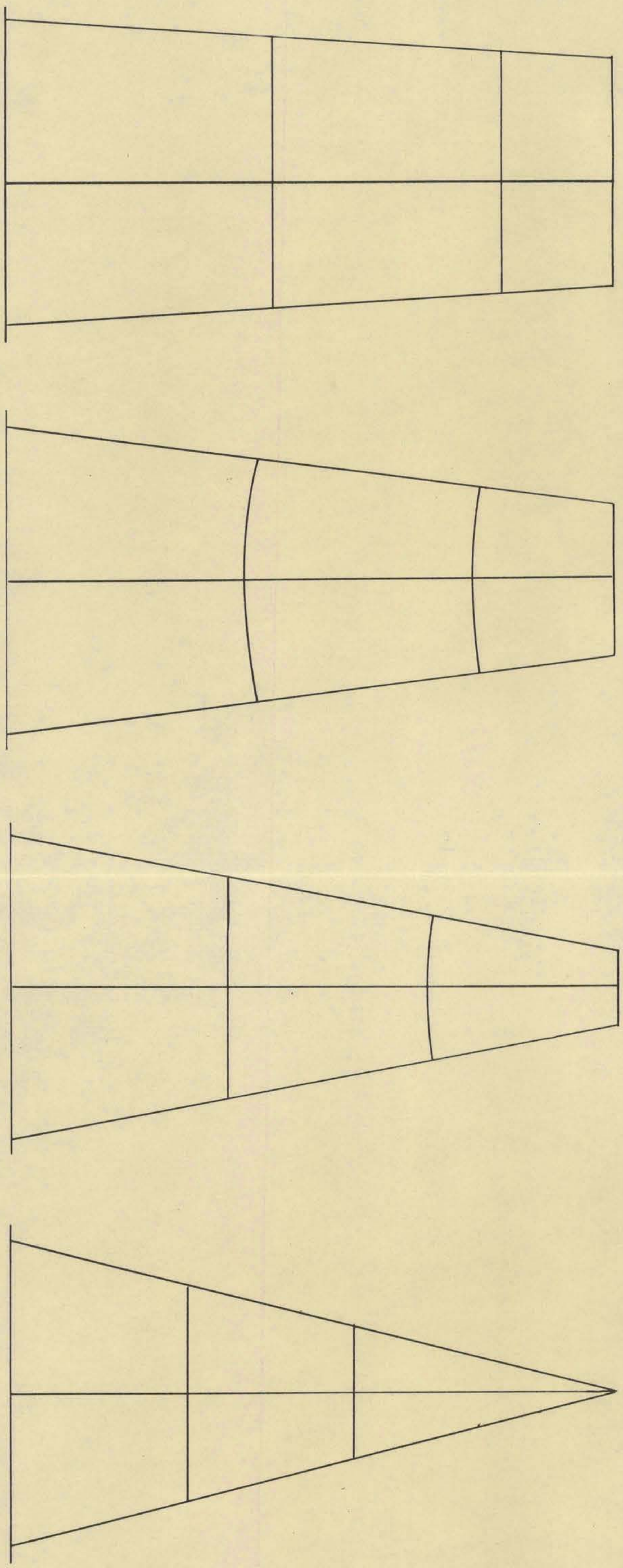


FIGURE 7.1 VARIATION OF NODAL PATTERN OF MODE 2/0 WITH ANGLE OF TAPER

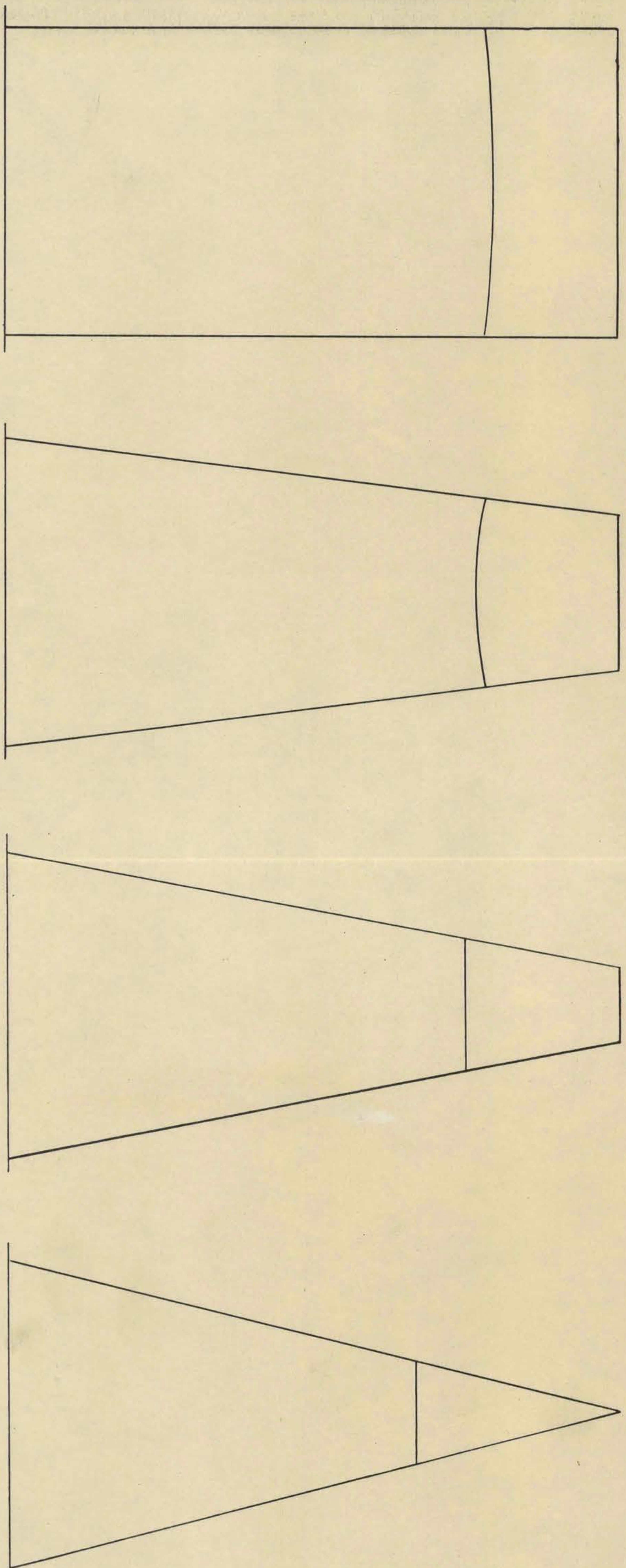


FIGURE 7.2 VARIATION OF NODAL PATTERN OF MODE 3/0 WITH ANGLE OF TAPER

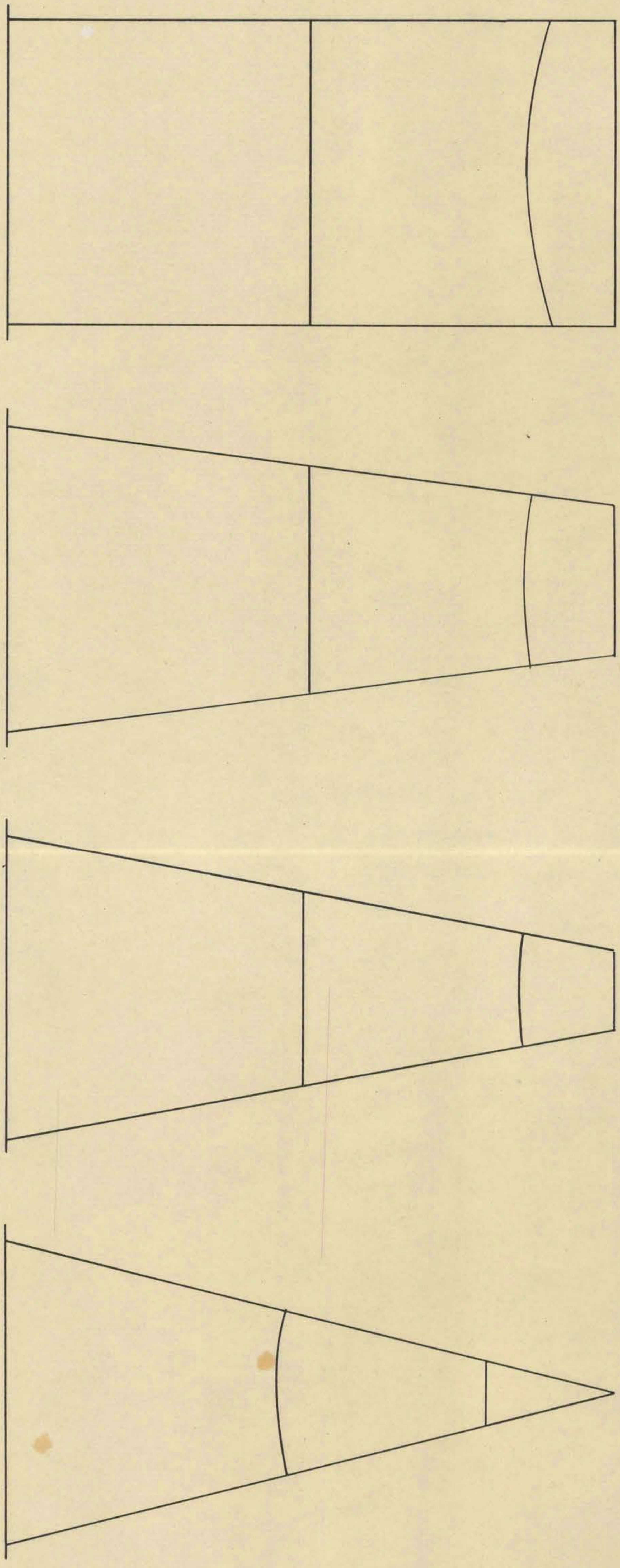
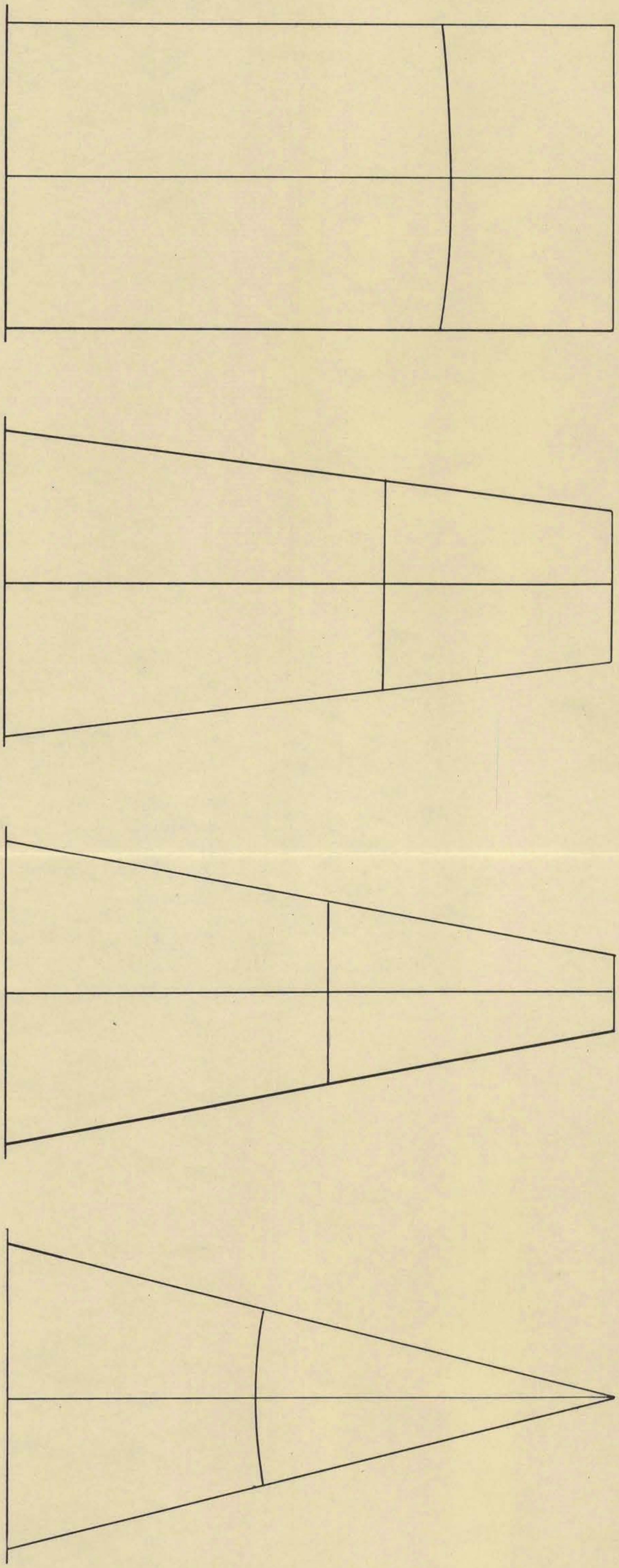


Figure 7.3 Variation of Nodal Pattern of Mode 2/1 with Angle of Taper



in position of the nodal lines as the angle of taper is increased. These expressions cannot, therefore, be used to calculate the natural frequencies of cantilevered tapered plates. Satisfactory results are obtained for modes 1/0 and 1/1 because there are no nodal lines parallel to the fixed edge of the plate. Any changes in the kinetic energy and strain energy of the vibrating plate at these modes is due to the change in area of the plate.

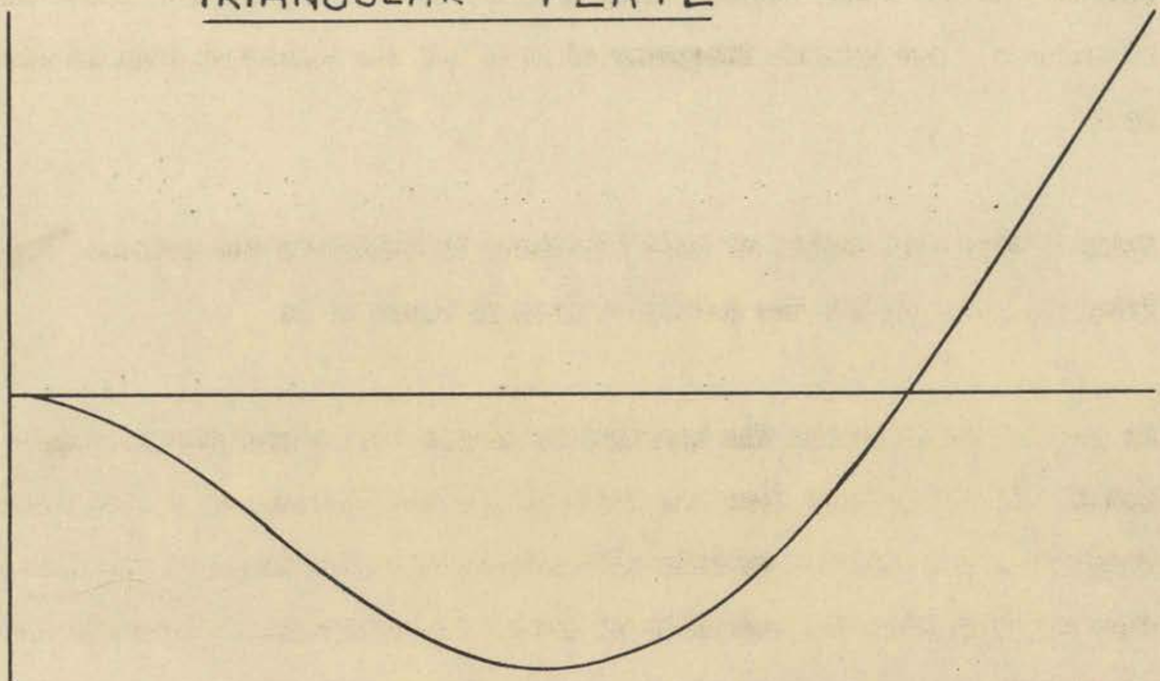
If the natural frequencies of the higher modes of tapered plates are to be calculated to any degree of accuracy using the Rayleigh-Ritz method the expression which is used to represent the deflected form must be one which predicts the position of the nodal lines and hence enables the maximum strain and kinetic energies of the plate to be calculated accurately. Natural frequencies using the Rayleigh-Ritz method are obtained from the expression

$$\omega^2 = \frac{U_{max}}{\frac{\rho h}{2g}\iint W^2 \, dy \, dx}$$

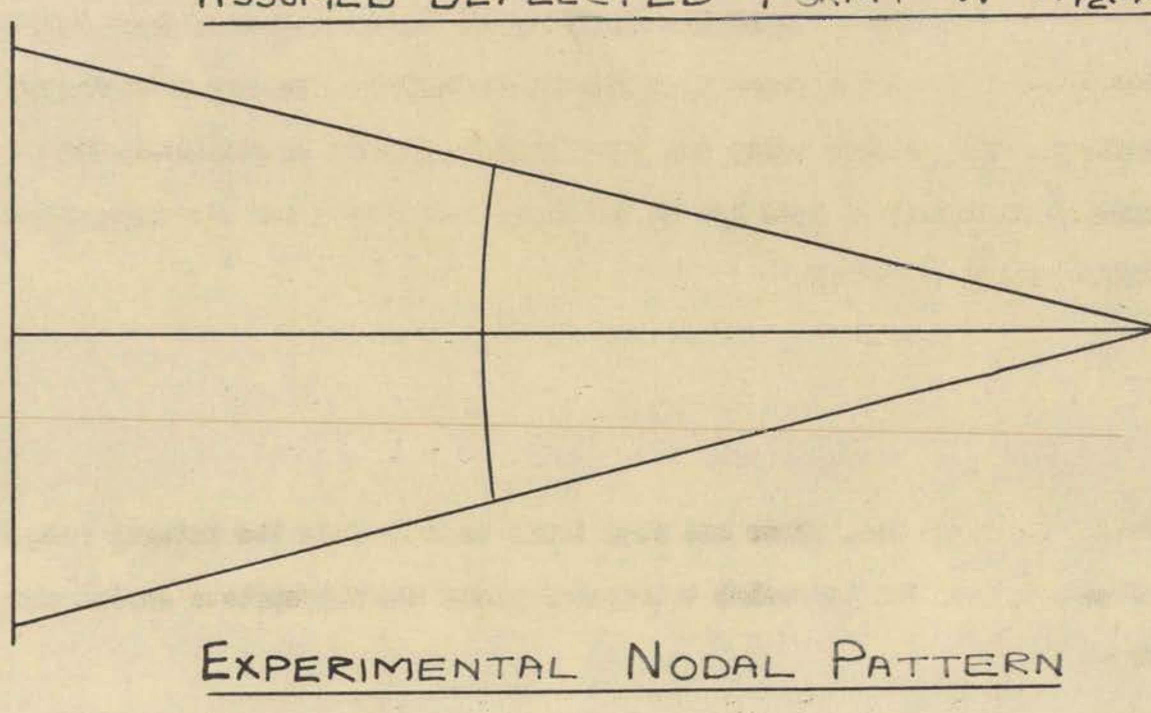
As the numerator and denominator depend on the assumed deflected form, they will both be altered by any change which is made in the assumed deflected form. It is, therefore, difficult to choose a deflected form which will give satisfactory results for the required frequencies. The reasonably close agreement between the calculated and experimental results does show, however, that the series which were used to represent the deflected form of the vibrating plate are in general satisfactory.

The results which are shown in sections 6.4.1. and 6.4.2. (pps 82-99) give the values of the coefficients of the terms which were used in each particular series and, therefore, indicate the relative importance of the terms in each series. Consider for example the natural frequencies of modes 2/1 and 3/1 of the isosceles triangular plates. When the single term expression of beam functions was used to calculate the natural frequencies.

FIGURE 7.5 COMPARISON OF EXPERIMENTAL AND ASSUMED NODAL PATTERNS OF MODE 2/1 OF ISOSCELES TRIANGULAR PLATE



ASSUMED DEFLECTED FORM :- $W = A\phi_2(x)\theta_1(y)$



EXPERIMENTAL NODAL PATTERN

of these modes the calculated values were respectively 37.2% and 42.4% lower than the experimental value (table 2, p.34) and as can be seen from figures 7.5 and 7.6 these deflected forms do not accurately predict the position of the nodal lines. Using a three term series of beam functions to calculate the natural frequency of mode 2/1 the deflected form is found to be

$$W = \left[-6.248\phi_1(x) + \underline{1.000\phi_2(x)} + 2.337\phi_3(x)\right]\theta_1(y)$$

Using a four term series of beam functions to calculate the natural frequency of mode 3/1 the deflected form is found to be

$$W = \left[-7.844\phi_1(x) + 3.304\phi_2(x) + \underline{1.000\phi_3(x)} - 3.047\phi_4(x)\right]\theta_1(y)$$

In each of these series the appropriate single term expression is underlined. It can be seen from the value of the coefficients of the additional terms that they exert a considerable influence on the shape of the deflected form and indicates the necessity of using these additional terms in the series for the calculation of the natural frequencies of tapered plates.

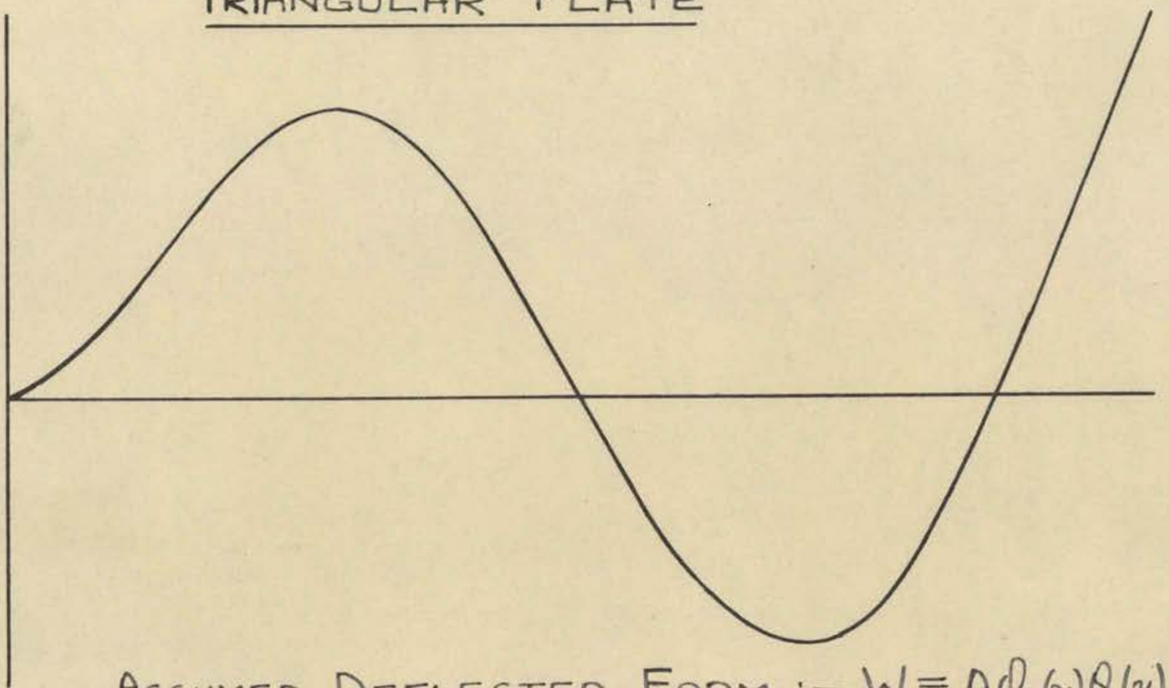
The value of the coefficients in the series of beam functions for modes m/0 and m/1 shows that for these families the series converges rapidly. For example using two three and four terms to calculate the natural frequency of mode 1/0 of the isosceles triangular the appropriate series are found to be

$$W = \left[1.000\phi_1(x) - 0.020\phi_2(x)\right]\theta_0(y)$$
$$W = \left[1.000\phi_1(x) - 0.014\phi_2(x) - 0.001\phi_3(x)\right]\theta_0(y)$$
$$W = \left[1.000\phi_1(x) - 0.014\phi_2(x) - 0.001\phi_3(x) + 0.000\phi_4(x)\right]\theta_0(y)$$

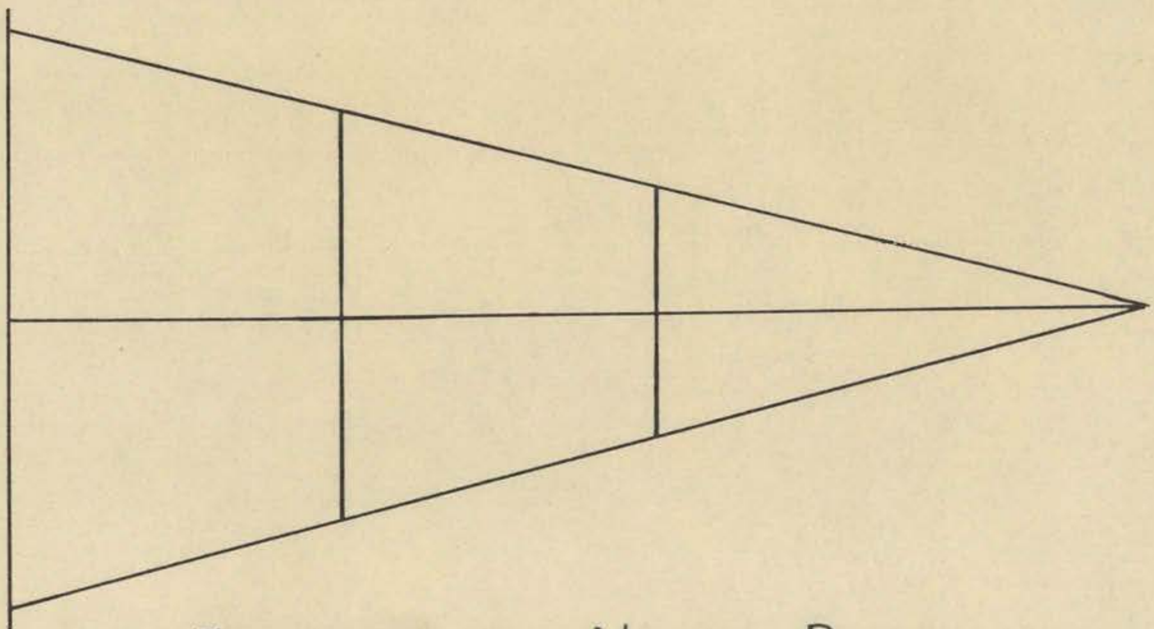
Using the first two, three and four terms to calculate the natural frequency of mode 2/0 of the isosceles triangular plate the appropriate series are found to be

$$W = \left[-0.770\phi_1(x) + 1.000\phi_2(x)\right]\theta_0(y)$$
$$W = \left[-0.775\phi_1(x) + 1.000\phi_2(x) - 0.029\phi_3(x)\right]\theta_0(y)$$
$$W = \left[-0.775\phi_1(x) + 1.000\phi_2(x) - 0.030\phi_3(x) - 0.001\phi_4(x)\right]\theta_0(y)$$

FIGURE 7.6 COMPARISON OF EXPERIMENTAL AND ASSUMED NODAL PATTERNS OF MODE 3/1 OF ISOSCELES TRIANGULAR PLATE

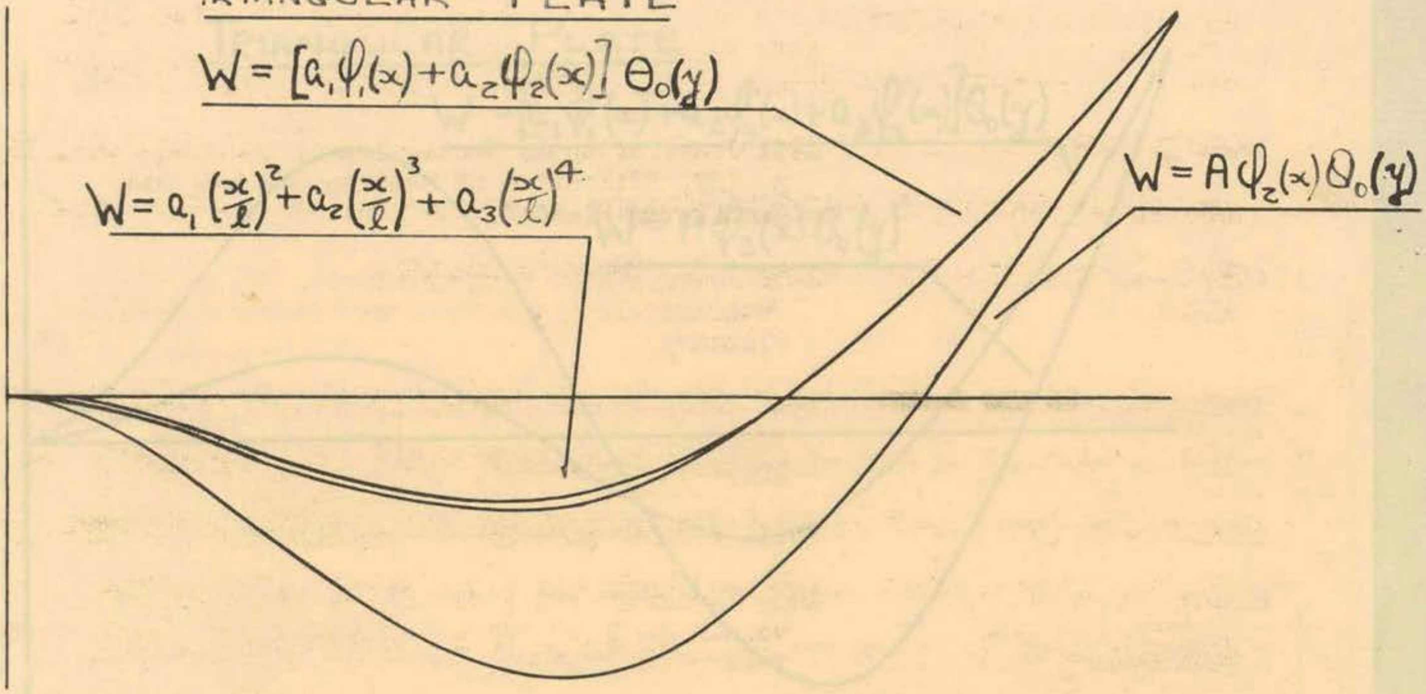


ASSUMED DEFLECTED FORM :- $W = A\phi_3(x)\theta_1(y)$

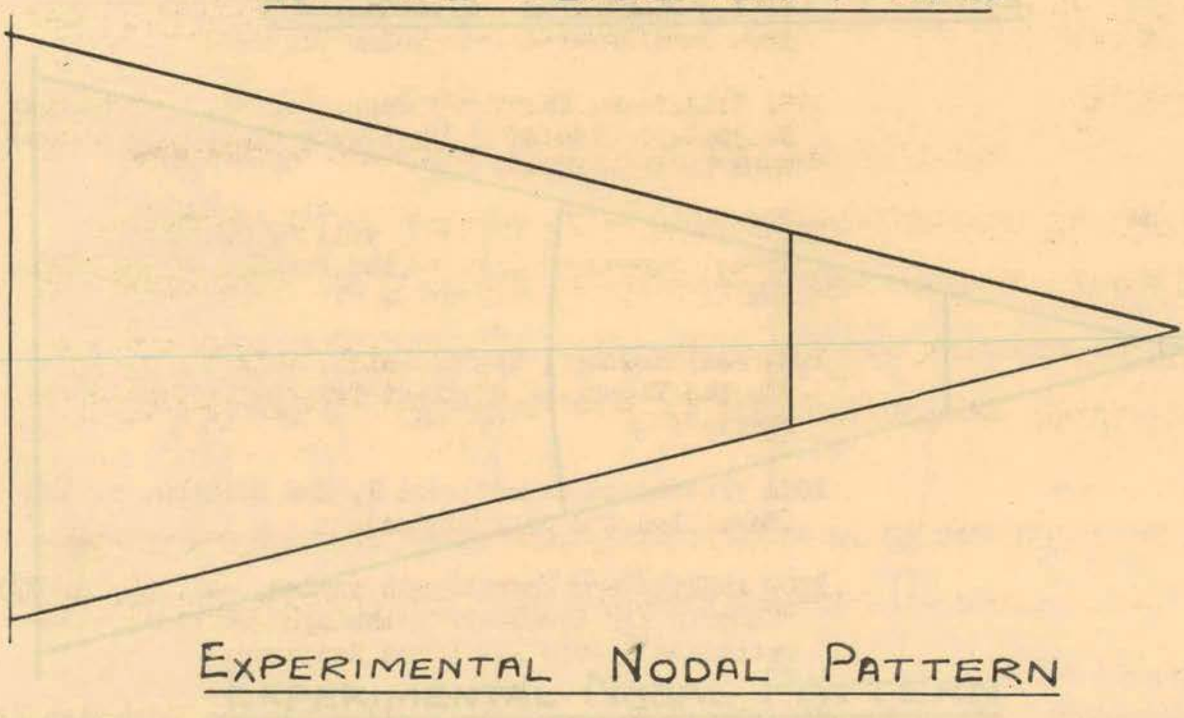


EXPERIMENTAL NODAL PATTERN

FIGURE 7.7 COMPARISON OF EXPERIMENTAL AND ASSUMED NODAL PATTERNS OF MODE 2/0 OF ISOSCELES TRIANGULAR PLATE



ASSUMED DEFLECTED FORMS



EXPERIMENTAL NODAL PATTERN

These series show that a point is reached when the addition of terms to the series, for the calculation of the natural frequency of any particular mode, exerts little influence on the shape of the deflected form and hence cannot improve to any appreciable extent the value of the natural frequency which was obtained using fewer terms. These results explain the reason for the natural frequency of each mode tending to a limit and why little improvement is obtained by the addition of terms to the series. It was also found that the use of a series of beam functions of the form,

$$W = [a_1\phi_1(x) + a_2\phi_2(x) + - - -]\theta_0(y)$$

to calculate the natural frequencies of the family m/0 of rectangular plates gave exactly the same results as the appropriate single term expression. On calculating the coefficients of the terms in the series it was found that the coefficients of the additional terms are zero for each mode. These extra terms do not therefore have any effect on the shape of the assumed deflected form and their inclusion in the series cannot, therefore, effect the value of the natural frequency which was obtained using the single term expression.

The close agreement between calculated and experimental results, the influence that the terms which were used in the series in addition to the single term expressions and the fairly rapid convergence of the series indicates that the series of beam functions which were used are satisfactory. It also means that the terms which were used of the series

$$W = \sum \sum A_{mn} \phi_m(x)\theta_n(y)$$

for the calculation of the natural frequencies of cantilevered rectangular and tapered plates are perhaps better than most other combinations that could have been used.

Figures 7.7, 7.8 and 7.9 are graphs of the deflected forms of modes 2/0, 3/0 and 2/1 of the isosceles triangular plate. These graphs show that the series of beam functions predicts the position of the nodal lines of the isosceles triangular plate much more accurately than the single term expression. They also show the close similarity of the deflected forms for these series and, therefore, indicates that the results which are obtained from a series of algebraic functions agree closely with those that are obtained from a series of algebraic functions.

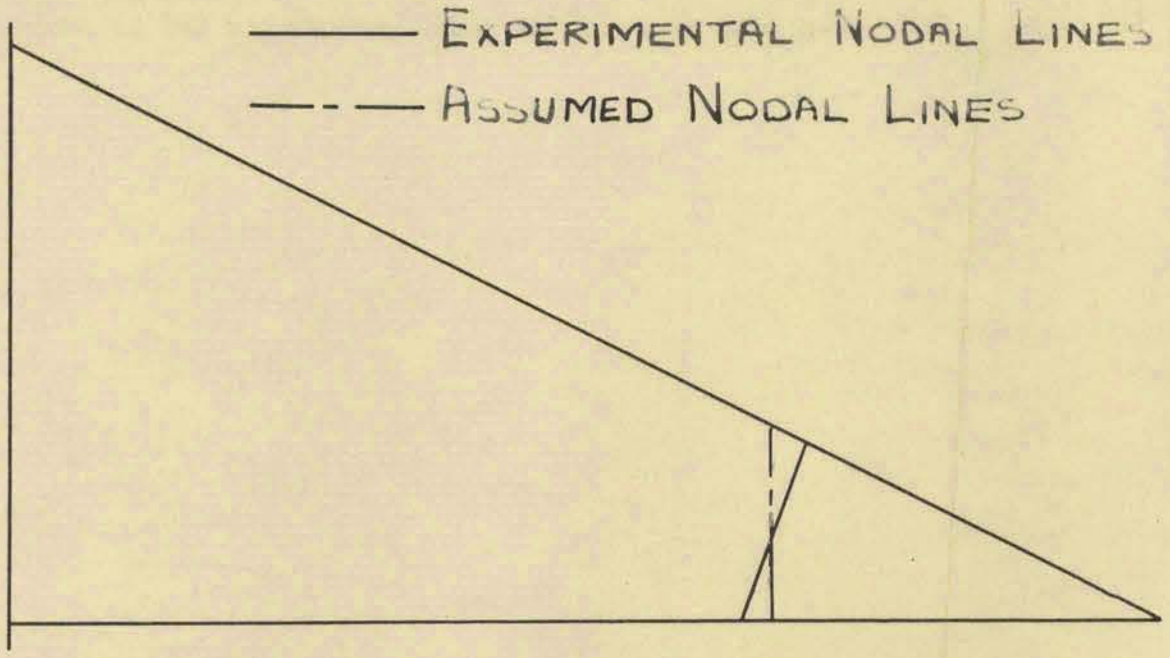
Although the results which were obtained for the natural frequencies are satisfactory the deflected forms which were used cannot give the true values of the natural frequencies of cantilevered plates. These deflected forms do not satisfy the boundary conditions at the free edges of the plate and are, therefore, not a completely accurate representation of the deflected form of the vibrating plate. The values of the natural frequencies which were obtained cannot, therefore, be completely accurate. As the angle of taper increases the deflected form becomes increasing more inaccurate, as the difference between the true boundary conditions of bending moment and shear force being zero and the assumed boundary conditions at the free edges increases. The effect on the calculated natural frequencies is that their accuracy relative to the experimental frequencies decreases as the angle of taper of the plate is increased. As the natural frequencies of the family m/1 of right-angled plates were calculated using oblique co-ordinates the difference between the assumed boundary conditions and the actual boundary conditions will be worse for these modes than for the corresponding modes of the symmetrical plates. The assumed deflected form becomes increasingly more inaccurate as the angle of taper increases, as the increase in angle between the oblique co-ordinates and the increase in the angle of taper of the plate makes the assumed boundary conditions less accurate as the area of the plate is reduced.

The assumed deflected forms assume that the nodal lines lie exactly parallel to the x and y axes. This assumption is almost correct for

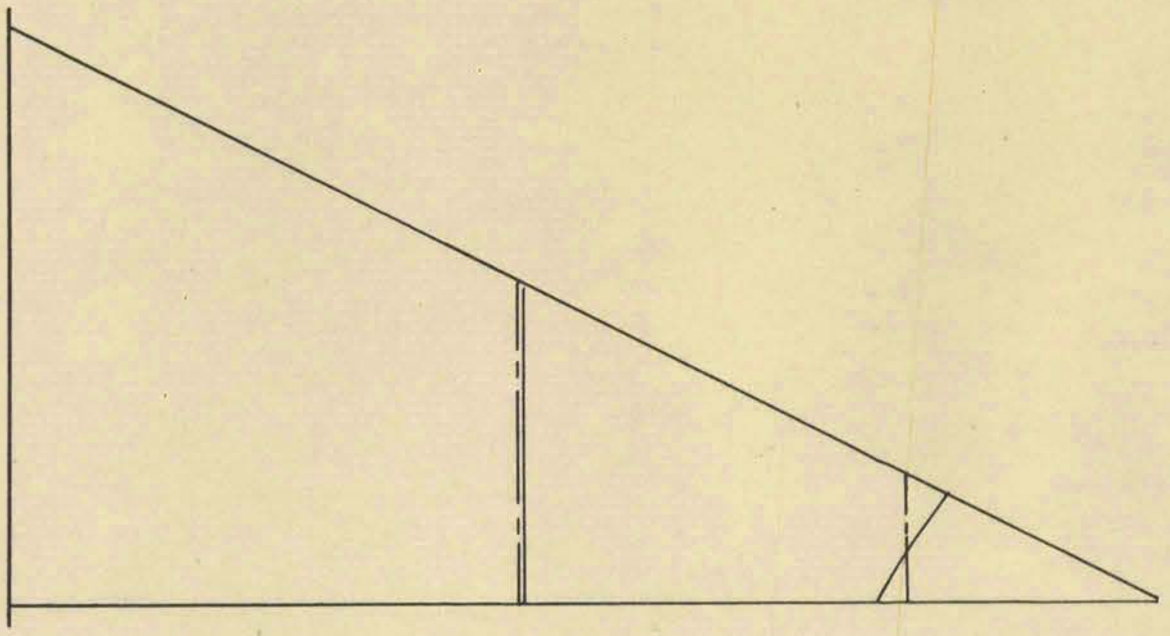
Figure 7.10 Comparison of Experimental and Assumed Nodal Patterns of Modes 2/0, 3/0, 1/1 and 2/1 of Right-Angled Triangular Plate

——— Experimental Nodal Lines

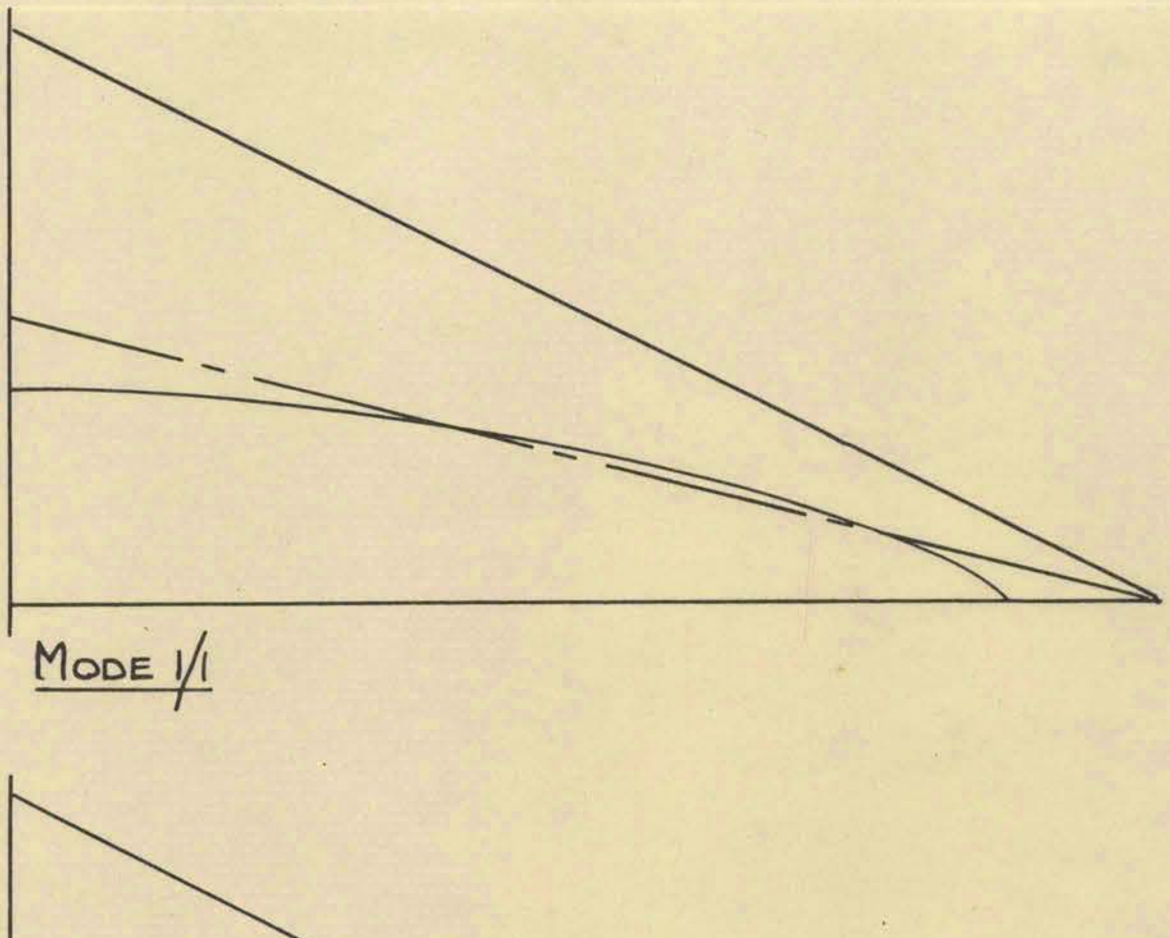
—-— Assumed Nodal Lines



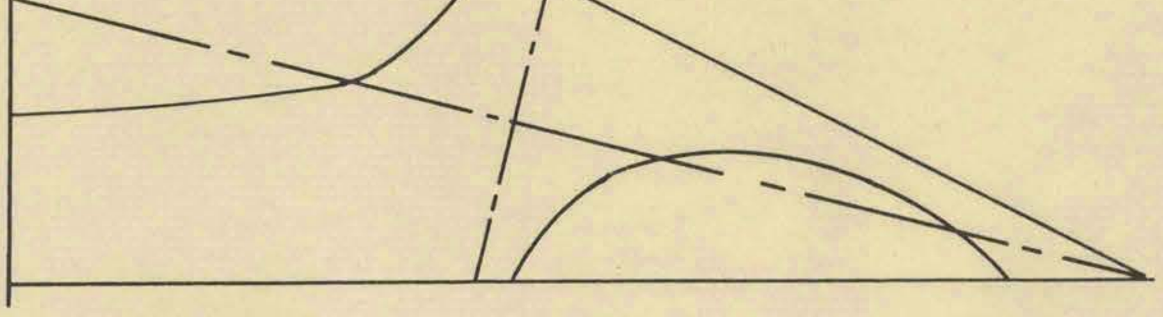
Mode 2/0



Mode 3/0



Mode 1/1



the symmetrical plates and any inaccuracy in the calculated value of the natural frequency of these plates due to the assumed shape of the nodal lines being different from their true shape will be negligible. For the right-angled plates, however, there is, in many cases, a considerable difference between the true shape of the nodal pattern and the assumed shape. In figure 7.10 the shape of the nodal patterns, which were found experimentally, for modes 2/0, 3/0, 2/1 and 3/1 of the right-angled triangular plate are shown. It can be seen that there is a considerable difference between the actual nodal pattern and the assumed nodal patterns. The difference is most marked for modes m/1. This together with the inaccuracy of the boundary conditions of the assumed deflected form probably accounts for the discrepancy between the calculated and experimental results of modes m/1 being larger for the right-angled plates than the symmetrical plates.

7.3 Comparison of Calculated Results with Results of Barton and Anderson

The calculated results which were obtained for the rectangular plate for modes 1/0, 2/0, 1/1 and 2/1 were compared with the calculated results which were obtained by Barton for these modes. The calculated frequencies of the rectangular plate of the symmetrical set which were calculated using four terms of the series of beam functions and four terms of the series of algebraic functions are shown along with Barton's values in table 30. It can be seen that Barton's values are slightly better than the results which were obtained in the present research programme. This is probably due to the fact that Barton used a nine term series of beam functions of the form

$$W = \sum\sum A_{mn}\,\phi_m(x)\,\theta_n(y)$$

and the assumed deflected form using that series gives a slightly more accurate representation of the nodal patterns of the modes under consideration than the series which was used in the present research programme. The extra work involved in using a series with a larger amount of terms is hardly worth the effort for the slight increase in accuracy which is obtained.

Table 30. Comparison of Calculated Natural Frequencies of Rectangular Plate with Barton's Results.

	Mode			
	1/0	2/0	1/1	2/1
Deflected Form	Natural Frequencies (c.p.s.)			
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_0(y)$	59.5	373		
$W = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$	59.5	373		
$W = [a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)]\theta_1(y)$			252	831
$W = (a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5)y$			257	856
Barton's Value	58.5	366	253	825
Experimental Value	58	359	256	820

The calculated results which were obtained for modes 1/0, 2/0, 1/1 and 2/1 of the isosceles triangular plate and modes 1/0 and 2/0 of the right-angled plate were compared with Anderson's values. It was found that the values for the modes 1/0 and 2/0 of the isosceles and right-angled triangular plates compared favourably with Anderson's values. Anderson's values for modes 1/1 and 2/1 of the isosceles triangular plate are, however, almost double those which were obtained in the present research programme. The calculated frequencies of the isosceles and right-angled triangular plates which were calculated using four terms of the series of beam functions and four terms of the algebraic series are shown along with Anderson's values in table 31.

Table 31 Comparison of Calculated Natural Frequencies of Isosceles and Right-angled Triangular Plates with Anderson's Results

a) Isosceles Triangular Plate

	Mode			
	1/0	2/0	1/1	2/1
Deflected Form	Natural Frequencies (c.p.s.)			
$W=[a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)+a_4\phi_4(x)]\theta_0(y)$	121	526		
$W=a_1x^2+a_2x^3+a_3x^4+a_4x^5$	121	526		
$W=[a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)+a_4\phi_4(x)]\theta_1(y)$			861	2137
$W=(a_1x^2+a_2x^3+a_3x^4+a_4x^5)y$			859	2142
Anderson's Value	120	520	1527	4370
Experimental Value	114	494	821	1994

b) Right-angled Plates

	Mode	
	1/0	2/0
Deflected Form	Natural Frequencies (c.p.s.)	
$W=[a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)+a_4\phi_4(x)]\theta_0(y)$	122	528
$W=a_1x^2+a_2x^3+a_3x^4+a_4x^5$	122	530
Anderson's Value	111	485
Experimental Value	112	477

7.4 Discussion of Difficulties in Calculating the Natural Frequencies of modes m/2 of Cantilevered Tapered Plates

During the course of the investigations the possibility of calculating the natural frequencies of modes m/2 of cantilevered trapezoidal and triangular plates, using the Rayleigh-Ritz method, was considered. The possibility of using either a series of beam functions or a series of algebraic functions to represent the deflected form of the vibrating plate was investigated. Considering first the series of beam functions, and assuming the deflected form of the vibrating plate to be similar to those which were used for the calculation of the natural frequencies of the families m/0 and m/1, it would be of the form

$$W = \left[a_1 \phi_1(x) + a_2 \phi_2(x) + \text{------}\right] \theta_2(y) \qquad 7.1$$

For the calculation of the natural frequencies of mode 1/2 the first term of the series could perhaps be used to represent the deflected form of the vibrating plate. It was found, however, that it leads to integrals of the type

$$\int_0^l \sinh \frac{2B_2}{b}(b - x\tan\theta)\left[\phi_1''(x)\right]^2 dx \qquad 7.2$$

when it is substituted in the expressions for the maximum kinetic energy and the maximum potential energy of the vibrating plate. The accurate evaluations of these integrals is impossible without using Simpson's rule on an electronic digital computer. One of these integrals was evaluated using autocode on a Ferranti Pegasus Computer and it was found that one hundred and twenty eight stations were required for its evaluation to obtain a sufficiently accurate result. The time required was between three and four minutes. There are sixty four integrals similar to equation 7.2 to be evaluated when the first term of equation 7.1 is used to represent the deflected form of the vibrating plate, and only the natural frequency of the first mode of the family would be obtained. If the higher modes of the family were required several terms of the series would be required to represent the deflected form of the vibrating plate. The number of integrals similar to equation 7.2 to be evaluated is considerably increased. The use of equation 7.1 for the calculation of the family m/2 of cantilevered tapered plates is, therefore, impractical.

One of the main difficulties in using a series of algebraic functions to calculate the natural frequencies of the family m/2 is the choice of a suitable series of functions to represent the deflected form of the vibrating plate. Before attempting to calculate the natural frequencies of the family of cantilevered tapered plates several deflected forms were used for the calculation of the natural frequencies of the mode 1/2 of cantilevered rectangular plates. The results, which were obtained, were compared with the experimental results. These results are shown in table 42 in appendix no. 6, and it can be seen that the frequency which was obtained from equation 8.23 agrees most closely with the experimental result. Equation 8.23 was used to calculate the natural frequency of mode 1/2 of the isosceles triangular plate and it was found that the calculated frequency was 12.8% lower than the experimental frequency. On the basis of these results, it can be concluded that equation 8.23 is suitable for the calculation of the natural frequency of mode 1/2 of rectangular plates, but is unsuitable for the calculation of the natural frequency of mode 1/2 of isosceles triangular plates. Further investigations are therefore required to obtain a suitable series of algebraic functions to calculate the natural frequencies of modes m/2 of cantilevered tapered plates.

7.5 Discussion of other Methods of Calculating the Natural Frequencies of Tapered Plates

The Rayleigh-Ritz method is not the only approximate method which may be used to calculate the natural frequencies of rectangular, trapezoidal and triangular plates. Several of the other methods which may be used are discussed in appendix no. 3. Like the Rayleigh-Ritz method, these methods lead to the solution of a series of algebraic equations of the type

$$\left.\begin{aligned} (c_{11}-\lambda K_{11})a_1 + --- + (c_{1n}-\lambda K_{1n})a_n &= 0 \\ - - - - - - - - - - - & \\ (c_{n1}-\lambda K_{n1})a_1 + --- + (c_{nn}-\lambda K_{nn})a_n &= 0 \end{aligned}\right\} \qquad 7.3$$

The natural frequencies are obtained by equating to zero the determinant of the coefficients a_m in equation 7.3. If a digital computer is not available, the size of the determinant which may be evaluated is limited to about the fourth order. As a result the use of the finite-difference method is more or less eliminated. The Galerkin and Kantorovich methods require that the assumed deflected form satisfies the boundary conditions. When a plate has free boundaries which are not parallel to either the x or y axis it is extremely difficult to obtain functions which satisfy the conditions at these boundaries and it would, therefore, be extremely difficult to calculate the natural frequencies of cantilevered tapered plates using either the Galerkin or Kantorovich method.

7.6 Discussion of Further Applications of the Methods of the Present Research Programme

In the present research programme experimental and calculated results were obtained for cantilevered, rectangular, trapezoidal and triangular plates. The ratio of the length to the fixed edge of the plate was two to one. The natural frequencies which were calculated successfully were several of those of the familes m/0 and m/1. At no point in the calculations were more than four terms of the appropriate series used for the calculation of the natural frequencies and only the first four natural frequencies of each family were calculated. Examination of those results and those of other investigators gives some indication of the extent, to which the present approach to the problem, may be applied to cantilevered plates having other length to fixed edge ratios and plates having other boundary conditions.

7.6.1. Discussion of extent of application to Cantilevered Tapered Plates.

The discrepancy between the calculated and experimental results of the modes which were considered increases as the angle of taper of the plate is increased. This is probably because the deflected forms which were used do not satisy the boundary conditions at the free edges and, therefore, become increasingly more inaccurate as the angle of taper is increased. The accuracy of the results would probably improve for plates having greater length to fixed edge ratios, as the angle of taper is reduced as the length to fixed edge ratio is increased.

The assumption that the nodal lines are parallel to the x and y axes is less accurate for right-angled plates than symmetrical plates and causes, therefore, further inaccuracies in the calculated frequencies of right-angled plates. Examination of the nodal patterns of modes 2/0 and 3/0 of right-angled triangular plates, which were obtained by Gustafson, Stokey and Zorowski, and which are shown in figure 7.11, shows that the assumption that the nodal lines of the family m/0 of these plates is less accurate for plates with low length to fixed edge ratios. As the length to fixed edge ratios of right-angled triangular plates increases the assumption

FIGURE 7.11 GUSTAFSON, STOKEY AND ZOROWSKI'S EXPERIMENTAL NODAL PATTERNS OF MODES 2/0 AND 3/0 OF RIGHT-ANGLED TRIANGULAR PLATES

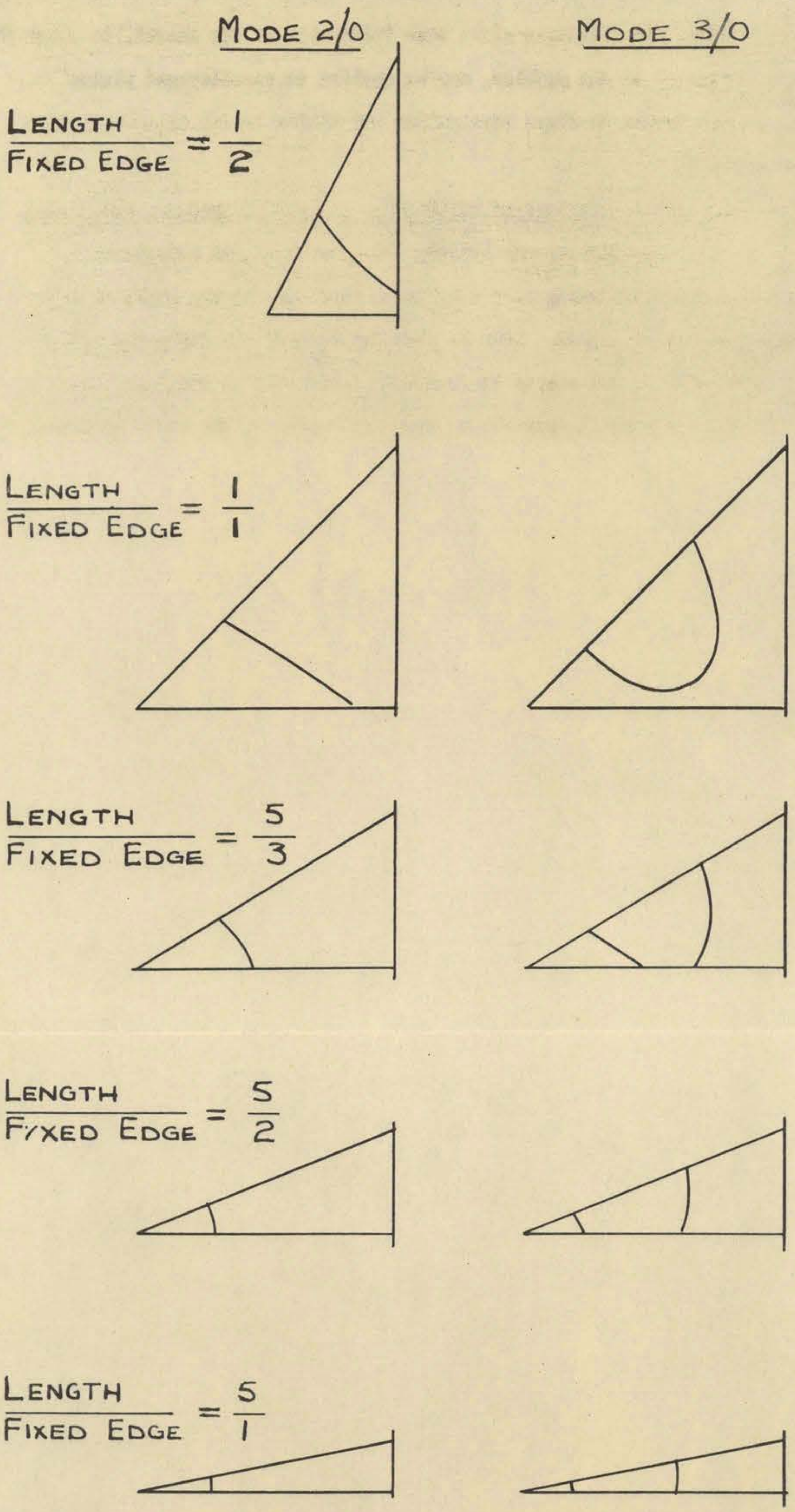
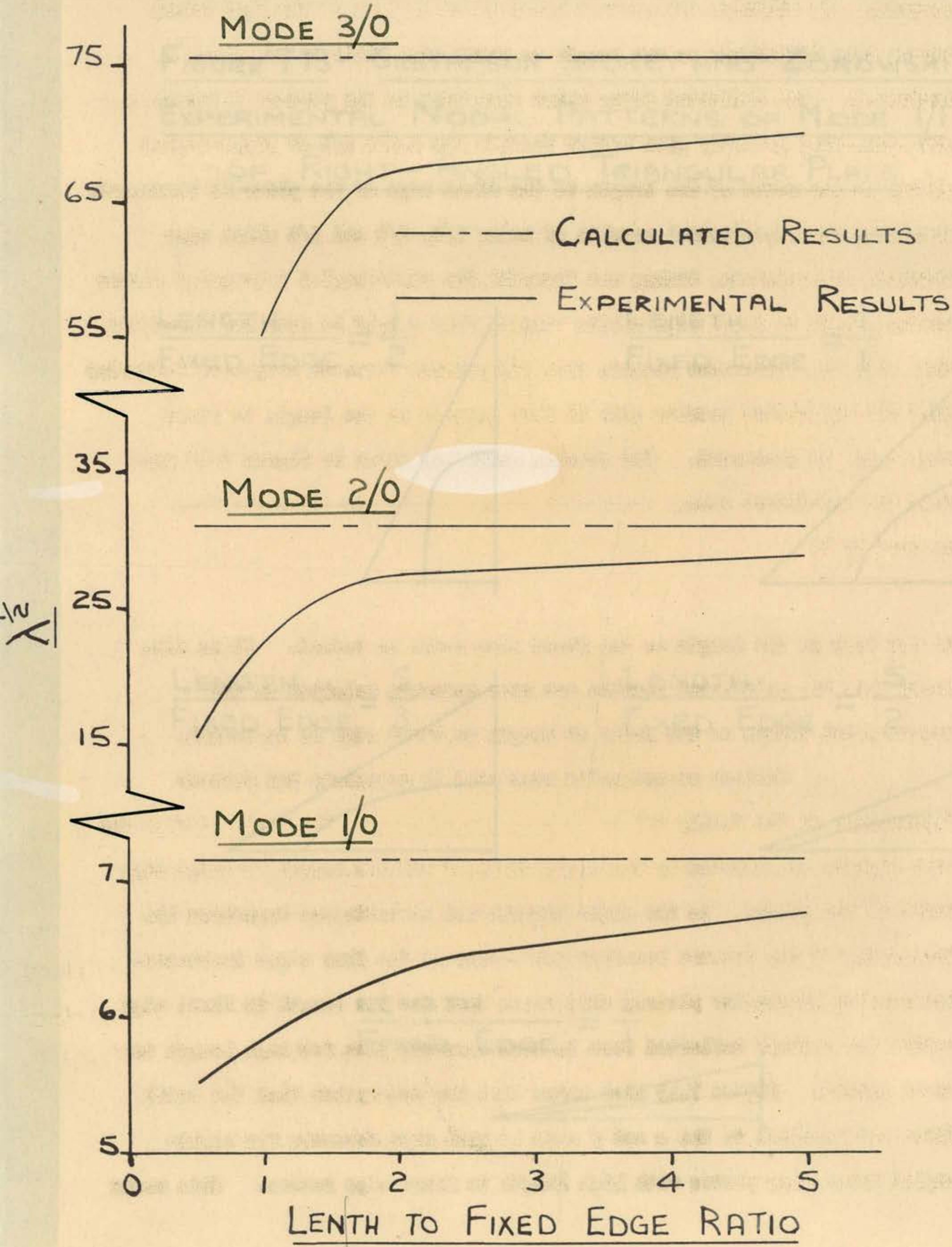


FIGURE 7.12 VARIATION OF $\lambda^{1/2}$ OF MODES M/0 OF RIGHT-ANGLED PLATES WITH LENGTH TO FIXED EDGE RATIO



that the nodal lines are parallel to the x and y axes becomes substantially correct. In addition the assumed boundary conditions at the free edges become less inaccurate as the length to fixed edge ratio of the plate is increased. The deflected forms which were used in the present research programme will probably give better results for modes m/0 of right-angled plates as the ratio of the length to the fixed edge of the plate is increased. Examining the experimental results of modes 1/0, 2/0 and 3/0 which were obtained by Gustafson, Stokey and Zorowski for right-angled triangular plates having length to fixed edge ratios varying from a half to five and comparing them with the calculated results from the present research programme indicates that the calculated results will in fact improve as the length to fixed edge ratio is increased. The results which are shown in figure 7.12 show that the calculated values which were obtained with the deflected form assumed to be

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x) + a_4 \phi_4(x)] \theta_0(y)$$

do not vary as the length to the fixed edge ratio is varied. It is also shown that the calculated results are more accurate relative to the experimental values as the ratio of length to fixed edge is increased.

Oblique co-ordinates were used to calculate the natural frequencies of the family m/1 of right-angled plates. The angle between the co-ordinates is affected by the angle of taper and the length to fixed edge ratio of the plate. As the angle between the co-ordinates increases the inaccuracy of the assumed boundary conditions at the free edges increases. Considering triangular plates, this means that for low length to fixed edge ratios the assumed deflected form is less accurate than for high length to fixed ratios. Figure 7.13 also shows that the assumption that the nodal lines are parallel to the x and y axis is much more accurate for right-angled triangular plates with high length to fixed edge ratios. This means that the use of oblique co-ordinates and the deflected forms, which were used in the present research project to calculate the natural frequencies of modes m/1 of right-angled plates would give better results for plates having high length to fixed edge ratios. The use of oblique co-ordinates would probably be inaccurate for the calculation of the natural frequencies of modes m/1 of triangular plates having swept back plan forms.

The number of terms which were used in any particular deflected form was limited by the amount of computational labour involved to evaluate the resulting determinant using a desk calculator. As the labour involved in evaluating a determinant of order greater than four is prohibitive, at no point in the calculations were more than four terms of the appropriate series used for the calculation of the natural frequencies of the families m/0 and m/1. The maximum amount of natural frequencies which could be obtained from any deflected form was, therefore, four. It would appear, however, from the results that reasonably accurate results could be obtained for more than four frequencies of each family if more terms were added to the series which represents the deflected form of the vibrating plate.

Due to the amount of computational labour involved it was considered impractical to use a series of the form

$$W = \left[a_1\phi_1(x) + a_2\phi_2(x) + - - -\right]\theta_n(y)$$

for the calculation of the natural frequencies of modes m/2, m/3 - - - - of cantilevered tapered plates. If the Rayleigh-Ritz method is to prove successful for the calculation of the natural frequencies of these modes and the labour involved is to be considerably reduced it would be better to express the deflection of the vibrating plate in the y direction by algebraic functions. For example for the calculation of the natural frequencies of modes m/2 and m/3 respectively, deflected forms of the form

$$W = [a_{11}\phi_1(x) + a_{12}\phi_2(x) + ---] + [a_{21}\phi_1(x) + a_{22}\phi_2(x) + ---]y^2$$

and

$$W = [a_{11}\phi_1(x) + a_{12}\phi_2(x) + ---]y + [a_{21}\phi_1(x) + a_{22}\phi_2(x) + ---]y^3$$

would probably be suitable.

7.6.2. Discussion of application to tapered plates having various boundary conditions.

In this research programme the only results to be obtained were for cantilevered plates. As the assumed deflected form must satisfy the geometric boundary conditions at simply supported and fixed edges, its accuracy will increase as the number of free edges is reduced. It should be possible, therefore, to extend directly, the methods which have been used for the calculation of the natural frequencies of cantilevered tapered plates to include tapered plates having other boundary conditions. If the boundaries in the y direction are free and the plate has any combination of boundary conditions in the x direction, a series of beam functions similar to those which were used for cantilevered plates could be used for the calculation of families m/0 and m/1. The assumed deflected forms for these modes would be series of the form

$$W = [a_1\phi_1(x) + a_2\phi_2(x) + - - -]\theta_0(y)$$

and

$$W = [a_1\phi_1(x) + a_2\phi_2(x) + - - -]\theta_1(y)$$

in which $\phi_1(x)$, $\phi_2(x)$ — — — — — — are the appropriate beam functions depending on the boundary conditions in the x direction and $\theta_0(y)$ and $\theta_1(y)$ are the appropriate functions of a beam which is free at each end. If beam functions were used to represent the deflected form of the vibrating plate in the y direction for the calculation of the natural frequencies of the familes m/2, m/3 - - - - - - , the computational labour would render the use of such a series impractical. For the calculation of the natural frequencies of the families m/2, m/3 - - - - - the series which represents the deflected form of the vibrating plate would, therefore, be beam functions in the x direction and algebraic functions in the y direction.

If it is required to calculate the natural frequencies of a tapered plate which is either simply supported or fixed at one or both of the y boundaries, the use of beam functions to represent the deflected form of the vibrating plate in the y direction leads to difficulties similar to those, which are encountered if beam functions are used to represent the deflected form of the vibrating plate in the y direction for the calculation of the natural frequencies of modes m/2 of cantilevered plates, Some form of algebraic function must be used, therefore, to represent the deflected in the y direction for the calculation of the natural frequencies of these plates. Suppose for example it is required to calculate the natural frequencies of a tapered plate which is simply supported on the y boundaries and has any boundary conditions on the x boundaries. By definition there are at least two nodal lines in the y directions. To calculate the natural frequencies of families m/2 and m/3 possible deflected forms are, therefore

$$W = [a_1\phi_1(x) + a_2\phi_2(x) + - - -][y - (c - x\tan\theta)][y - (x\tan\theta - c)]$$

$$W = [a_1\phi_1(x) + a_2\phi_2(x) + - - -][y - (c - x\tan\theta)][y - (x\tan\theta - c)]y$$

Similar deflected forms can be constructed for the calculation of the natural frequencies of modes m/1 and m/2 of tapered plates having one of the y boundaries free and the other simply supported. There is, however, some difficulty in choosing a suitable deflected form for the calculation of the natural frequencies of the families m/4, m/5 - - - - - of plates which are simply supported at both y boundaries and of the families m/3, m/4 - - - - of plates which are simply supported at one y boundary and free at the other. Further investigation would be required to obtain a suitable deflected form for the calculation of the natural frequencies of these modes.

It is required to calculate the natural frequencies of the families m/2 and m/3 of a tapered plate which is fixed at both boundaries in the y direction and having any conditions at the boundaries in the x direction possible deflected forms for the vibrating plate would be,

$$W=[a_1\phi_1(x)+a_2\phi_2(x)+ - -]\{[y-(c-x\tan\theta)][y-(x\tan\theta-c)]\}^2$$

and

$$W=[a_1\phi_1(x)+a_2\phi_2(x)+ - - -]\{[y-(c-x\tan\theta)][y-(x\tan\theta-c)]\}^2 y$$

Similar deflected forms can be constructed for the calculation of the natural frequencies of familes m/2 and m/3 having any combination of fixed and simply supported boundaries in the y direction and for the calculation of the natural frequencies of plates having one boundary fixed and one boundary free in the y direction. As before further investigations are required for the choice of suitable deflected forms for the calculation of the natural frequencies of the families m/4, m/5 - - - - of plates which are fixed at both boundaries in the y direction and of the families m/3, m/4 - - - - - of plates which are fixed at one boundary and free at the other in the y direction.

7.7 Conclusions

In conclusion the main findings and conclusions of the research project can be summarized as follows,

1) The method which was used successfully by Warburton for the calculation of the natural frequencies of all modes of vibration of rectangular plates was found to be unsatisfactory for the calculation of the natural frequencies of cantilevered tapered plates and would probably be unsatisfactory for the calculation of the natural frequencies of tapered plates having other boundary conditions.

2) The Rayleigh-Ritz method assuming the deflected form of the vibrating plate to be either a series of beam functions of the form

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + \cdots] \theta_0(y)$$

or a series of algebraic functions of the form

$$W = a_1 x^2 + a_2 x^3 + \cdots$$

has proved successful for the calculation of the natural frequencies of the family m/0 of cantilevered tapered plates. The method can also be used for the calculation of the natural frequencies of the family m/1 of these plates if the deflected form of the vibrating plate is assumed to be either a series of beam frequencies of the form

$$W = [a_1 \phi_1(x) + a_2 \phi_2(x) + \cdots] \theta_1(y)$$

or a series of algebraic functions of the form

$$W = (a_1 x^2 + a_2 x^3 + \cdots) y$$

The good agreement between the calculated and experimental results suggests that the choice of a deflected form from which only the natural frequencies of one family are obtained, is the most suitable choice for cantilevered tapered plates. The natural frequencies of the corresponding modes of tapered plates with other boundary conditions could probably be calculated using the Rayleigh-Ritz method with similar deflected forms.

3) The Rayleigh-Ritz method, using the same deflected forms would probably be more accurate for the calculation of the natural frequencies of the families m/0 and m/1 of cantilevered tapered plates having greater length to fixed edge ratios than the plates which were considered in the present research programme. The method would probably be more accurate for tapered plates having fewer free edges but less accurate for cantilevered trapezoidal and triangular plates of swept back plan form.

CHAPTER VIII

APPENDICES

Appendix No. 1 — **Notation**

a, c, d	Dimensions used in calculation of natural frequencies of triangular plates.
g	Acceleration due to gravity.
h	Thickness of plate.
l, b	Length and breadth of rectangular plate
m, n	Number of nodal lines in x and y directions
n, t, x, y	Coordinate distances in plane of plate
u	Arbitrary dependent variable in x and y
z	Width of mesh in finite-difference expression
A, B	Constants in approximate solution of differential equation
D =	$\dfrac{Eh^3}{12(1-\mu^2)}$
E	Young's modulus
$G_x\ H_x\ J_x$, $G_y\ H_y\ J_y$	Functions in frequency expression
L	Arbitrary differential operator
T max	Maximum kinetic energy of vibrating plate
U max	Maximum potential energy of vibrating plate
W	Maximum transverse displacement of a point
$a_1, a_2 \ldots a_n$	Constants in approximate solution of differential equation
$c_{11}, c_{12} \ldots c_{nn}$, $k_{11}, k_{12} \ldots k_{nn}$	Constants in simultaneous algebraic equations depending on differential equation which is being solved.
$f_1(x), f_2(x) \ldots f_n(x)$	Functions, satisfying boundary conditions in the x direction, which are obtained when Kantorovich's

method is used to obtain an approximate solution to a partial differential equation.

$f_1(x,y), f_2(x,y) \ldots f_n(x,y)$ Functions chosen to satisfy boundary conditions in the x and y directions when using Galerkin's method to obtain an approximate solution of a partial differential equation

$g_1(y), g_2(y) \ldots g_n(y)$ Functions chosen to satisfy boundary conditions in the y direction when using Kantorovich's method to obtain an approximate solution of a partial differential equation.

x_1, x_2, y_1, y_2 Plate boundaries

$A_{00}, A_{11} \ldots A_{nn}$ Constants in approximate solution of differential equation

$W_0, W_2 \ldots W_{12}$ Deflections, at points of intersection of grid lines, in finite-difference expression

$\alpha, \beta, \varphi, \theta$ Angles used in calculation of natural frequencies of triangular plates.

$$\lambda = \frac{\rho h \omega^2 l^4}{g D}$$

$$\lambda_1^2 = \frac{\rho h \omega^2 l^4}{g D \pi^4}$$

ρ Density of plate

μ Poisson's ratio

ω Natural frequency

$\alpha_1 : \alpha_2 - - - \alpha_n$ Constants in polynomial depending on differential equation which is being solved.

$\alpha_m, \beta_m, \gamma_m, \varepsilon_m$ Constants in expressions which represent the normal modes of vibration of uniform beams.

$\left.\begin{matrix}\varphi_1(x), \varphi_2(x) - - \\ \theta_0(y), \theta_1(y) - -\end{matrix}\right\}$ Expressions which represent the normal modes of vibration of a uniform beam.

Appendix No. 2: Values of α and ε and integrals of characteristic Functions of a uniform cantilevered beam.

Table 32: Values of α and ε

m	α	ε
1	0.734095	1.875104
2	1.018466	4.694091
3	0.999224	7.854757
4	1.000035	10.995540
5	0.999998	14.137168
$m>5$	1.0	$(2m+1)\pi/2$

The functions $\varphi_1(x), \varphi_2(x) - -$, which represent the normal modes of vibration of a uniform cantilevered beam are orthogonal in the interval 0 to ℓ. That is, for any two functions $\varphi_m(x)$ and $\varphi_n(x)$ in the set, the following relationships hold

$$\int_0^{\ell} \varphi_m(x)\,\varphi_n(x)\,dx = \ell \quad (\text{for } m = n)$$
$$= 0 \quad (\text{for } m \neq n)$$

The second derivatives of the functions are also orthogonal and satisfy the relationships

$$\int_0^{\ell} \varphi_m''(x)\,\varphi_n''(x)\,dx = \varepsilon_m^4/\ell^3 \quad (\text{for } m = n)$$
$$= 0 \quad (\text{for } m \neq n)$$

Table 33: Values of $\frac{1}{\ell^2}\int_0^{\ell} x\,\varphi_m(x)\,\varphi_n(x)\,dx$

$n \backslash m$	1	2	3	4
1	0.806537	-0.153516	0.020323	-0.008787
2	-0.153516	0.594149	-0.190817	0.020584
3	0.020323	-0.190817	0.532366	-0.196971
4	-0.008787	0.020584	-0.196971	0.516543

Table 34: Values of $\frac{1}{l^3}\int_0^l x^2 \phi_m(x)\,\phi_n(x)\,dx$

$n \backslash m$	1	2	3	4
1	0.806537	- 0.153516	0.020323	- 0.008787
2	- 0.153516	0.594149	- 0.190817	0.020584
3	0.020323	- 0.190817	0.532366	- 0.196971
4	- 0.008787	0.020584	- 0.196971	0.516543

Table 35: Value of $\frac{1}{l^4}\int_0^l x^3 \phi_m(x)\,\phi_n(x)\,dx$

$n \backslash m$	1	2	3	4
1	0.579140	- 0.245557	0.090729	- 0.034206
2	- 0.245557	0.323966	- 0.215549	0.093940
3	0.090729	- 0.215549	0.278709	- 0.199004
4	- 0.034206	0.093940	- 0.199004	0.265636

Table 36: Value of $l\int_0^l \phi_m'(x)\,\phi_n'(x)\,dx$

$n \backslash m$	1	2	3	4
1	4.647771	- 7.379862	3.941504	- 6.593380
2	- 7.379862	32.417317	-22.352419	13.582447
3	3.941504	-22.352419	77.298764	-35.648186
4	- 6.593380	13.582447	-35.648186	142.901571

Table 37: Value of $\int_0^l x\,\phi_m'(x)\,\phi_n'(x)\,dx$

$n \backslash m$	1	2	3	4
1	3.076892	- 6.957547	5.013591	- 5.720238
2	- 6.957547	23.770190	-24.232361	17.225832
3	5.013591	-24.242461	52.346686	-43.986517
4	- 5.720238	17.225832	-43.986517	91.442579

Table 38: Value of $l^2\int_0^l x\,\varphi_m''(x)\,\varphi_n''(x)\,dx$

$n \backslash m$	1	2	3	4
1	2.391647	-11.893436	- 4.408633	- 3.735291
2	-11.893436	197.047784	- 259.409317	-54.836016
3	- 4.408633	- 259.409317	1780.066222	-1469.267649
4	- 3.735291	-54. 836016	-1469.267649	7066.806489

Table 39: Value of $l\int_0^l x^2\,\varphi_m''(x)\,\varphi_n''(x)\,dx$

$n \backslash m$	1	2	3	4
1	0.760371	- 6.806442	3.838763	0.859112
2	- 6.806442	109.480593	-216.285680	64.562323
3	3.838763	-216.285680	1122.691066	-1344.883663
4	0.859112	64.562323	-1344.883663	4581.139864

Table 40: Value of $\int_0^l x^3\,\varphi_m''(x)\,\varphi_n''(x)\,dx$

$n \backslash m$	1	2	3	4
1	0.308997	- 3.763202	5.060720	- 0.756240
2	- 3.763202	65.526095	-163.659373	107.943089
3	5.060720	-163.659373	773.493821	-1111.273009
4	- 0.756240	107.943089	-1111.273009	3277. 367157

Appendix No. 3: Solution of differential equations by approximate methods

The exact solution of many differential equations is extremely difficult. When difficulties arise in obtaining the exact solution of a differential equation it is often necessary to use a method which gives an approximate solution. Many approximate methods of solving differential equations have been developed and several of these methods will be described briefly in this section.

1. Galerkin's Method

The method proposed by Galerkin (1915) may be used to obtain an approximate solution of a complicated differential equation.

Consider, for example, a differential equation of the type

$$L(u) = 0 \qquad 8.1$$

where L is some differential operator in two variables. If it proves difficult to obtain the exact solution of equation 8.1, Galerkin's method may be used to obtain an approximate solution of the form

$$w = a_1 f_1(x,y) + a_2 f_2(x,y) + \cdots + a_n f_n(x,y) \qquad 8.2$$

in which each of the functions $f_m(x,y)$ must satisfy the boundary conditions of the problem. The approximate solution, equation 8.2, is substituted in equation 8.1 which is then multiplied by each of the functions $f_m(x,y)$. Each product is integrated between the required limits and the integral equated to zero giving a series of equations of the type

$$\iint \left\{ L\left[a_1 f_1(x,y) + \cdots + a_n f_n(x,y)\right] \right\} f_m(x,y)\, dy\, dx = 0 \qquad 8.3$$

from which the coefficients a_m can be determined. These are substituted in equation 8.2 to give the approximate solution of equation 8.1.

2. Kantorovitch's Method

To obtain an approximate solution of a differential equation using the method proposed by Kantorovich (1933), it must be replaced by the corresponding variational problem. Consider for example the solution of the following differential equation,

$$L(u) = 0 \tag{8.4}$$

where L is a differential operator in two variables. The corresponding variational problem is of the form,

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} F\left(u, x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) dy\, dx = 0 \tag{8.5}$$

The approximate solution of the problem is assumed to be of the form

$$u = g_1(y) f_1(x) + g_2(y) f_2(x) + \text{---} g_n(x) f_n(x) \tag{8.6}$$

in which the functions $g_m(y)$ satisfy the boundary conditions at y_1 and y_2. Equation 8.6 is substituted in equation 8.5 and integrated with respect to y. The original problem of obtaining an expression which gives a double integral a minimum value, is thus reduced to the problem of finding an expression which gives a single integral of the form

$$\int_{x_1}^{x_2} F\left(x, f_m(x), f_m'(x)\right) dx = 0 \tag{8.7}$$

a minimum value. Using variational calculus to obtain the expression which gives equation 8.7 a minimum value, the problem reduces to the solution of a system or ordinary differential equations to obtain the functions $f_m(x)$ satisfying the boundary conditions at x_1 and x_2. These functions, $f_m(x)$, are substituted in equation 8.6 to obtain the approximate solution of equation 8.4.

3. Finite-Difference Method

The solution of a differential equation of the type

$$L(u) = 0$$

using the finite-difference method is not obtained as a continuous function. The values of u are obtained at a finite number of points within a specified region for which the boundary conditions must be known. At each of an arbitrary number of points within the region, for which the solution is required, the derivatives in the differential equation are replaced by expressions in terms of the unknown u at the particular point under consideration and of the unknown u at neighbouring points. The original differential equation is replaced, therefore, by a series of simultaneous, algebraic equations which can be solved to obtain the approximate value of u at each point.

Appendix No. 4: Solution of the equation of motion of a vibrating plate by approximate methods

One of the differential equations, which are encountered in engineering and of which the exact solution is difficult, is that of the vibration of thin, flat plates, of uniform thickness. The solution of that equation depends on the shape of the plate and the boundary conditions of the plate, and for most plate shapes and boundary conditions it is necessary to use an approximate method to obtain a solution. This section will be devoted, therefore, to discussing the application of the approximate methods which were described in Appendix No. 3, to the solution of the equation of motion of vibrating cantilevered, rectangular, trapezoidal and triangular shaped plates.

1. Galerkin's Method

If Galerkin's method is used to solve the equation of motion of a vibrating plate, equation 3.1, each of the functions $f_m(x,y)$ in the approximate solution, equation 8.2, must satisfy the geometric and the natural boundary conditions of the plate. At a boundary which is parallel to the y axis, these are, at a fixed edge

$$W = 0 \qquad \text{and} \qquad \frac{\partial W}{\partial x} = 0$$

at a simply supported edge

$$W = 0 \qquad \text{and} \qquad \left(\frac{\partial^2 W}{\partial x^2} + \mu \frac{\partial^2 W}{\partial y^2}\right) = 0$$

at a free edge

$$\left(\frac{\partial^2 W}{\partial x^2} + \mu \frac{\partial^2 W}{\partial y^2}\right) = 0 \qquad \text{and} \qquad \left(\frac{\partial^3 W}{\partial x^3} + (2-\mu) \frac{\partial^3 W}{\partial x \partial y^2}\right) = 0$$

When a rectangular plate has any combination of fixed and simply supported edges it is possible to construct a series of simple functions which satisfy all the boundary conditions and it would be possible, therefore, to calculate its natural frequencies using Galerkin's method. When a rectangular plate has free boundaries it is much more difficult to construct a series of functions which satisfy the boundary conditions.

It is even more difficult to construct a series of functions which satisfy the boundary conditions when the plate has free boundaries which are not parallel to either the x or y axis. Consider, for example, the solution of equation 3.1 for the case of a trapezoidal cantilevered plate fixed at x = 0 and free at the other three edges. (Fig. 8.1)

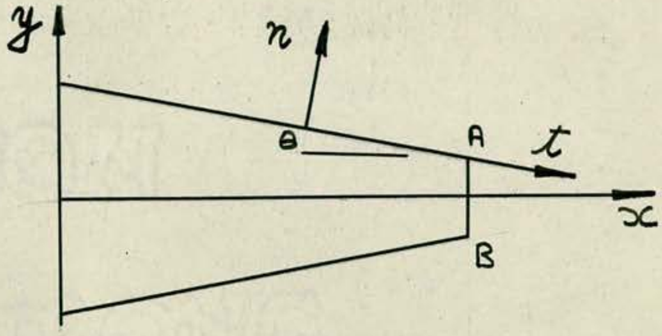


Fig. 8.1

At the sloping edges

$$\left(\frac{\partial^2 W}{\partial n^2} + \mu \frac{\partial^2 W}{\partial t^2}\right) = 0 \qquad 8.8$$

$$\left(\frac{\partial^3 W}{\partial n^3} + (2-\mu)\frac{\partial^3 W}{\partial n \partial t^2}\right) = 0 \qquad 8.9$$

In addition the condition

$$\left(\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 W}{\partial n \partial t}\right) = 0 \qquad 8.10$$

at the free corners A and B must be satisfied.

Equations 8.8, 8.9 and 8.10 become in terms of derivatives in the x and y directions,

$$\left[\left(\frac{\partial^2 W}{\partial x^2}\sin^2\theta - \frac{\partial^2 W}{\partial x \partial y}\sin 2\theta + \frac{\partial^2 W}{\partial y^2}\cos^2\theta\right) + \mu\left(\frac{\partial^2 W}{\partial x^2}\cos^2\theta + \frac{\partial^2 W}{\partial x \partial y}\sin 2\theta + \frac{\partial^2 W}{\partial y^2}\sin^2\theta\right)\right] = 0 \qquad 8.11$$

$$\left\{\left[\frac{\partial^3 W}{\partial x^3}\sin^3\theta - \frac{\partial^3 W}{\partial x^2\partial y}\left(\sin\theta\sin 2\theta+\sin^2\theta\cos\theta\right) + \frac{\partial^3 W}{\partial x\partial y^2}\left(\sin\theta\cos^2\theta+\sin\theta\sin 2\theta\right) - \frac{\partial^3 W}{\partial y^3}\cos^3\theta\right] + 2(1-\mu)\left[\frac{\partial^3 W}{\partial x^3}\sin\theta\cos^2\theta + \frac{\partial^3 W}{\partial x^2\partial y}\left(\sin\theta\sin 2\theta - \cos^3\theta\right) + \frac{\partial^3 W}{\partial x\partial y^2}\left(\sin^3\theta - \sin 2\theta\cos\theta\right) + \frac{\partial^3 W}{\partial y^3}\sin^2\theta\cos\theta\right]\right\} = 0 \qquad 8.12$$

$$\left[\frac{1}{2}\left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2}\right)\sin 2\theta + \frac{\partial^2 W}{\partial x\partial y}\left(1+\cos 2\theta\right)\right] = 0 \qquad 8.13$$

As it would be extremely difficult to obtain functions which satisfy the boundary conditions of cantilevered, trapezoidal plates, it would be impractical to use Galerkin's method to calculate their natural frequencies.

2. Kantorovich's Method

If Kantorovich's method is used to obtain a solution of equation 3.1 the functions $g_m(y)$ in the expression for the assumed deflected form, equation 8.6, must satisfy the geometric and the natural conditions on the y boundaries. The method could be used, therefore, to calculate the natural frequencies of a rectangular plate with the y boundaries having any combination of fixed and simply supported edges, as it would be a relatively simple matter to construct a series for the deflected form in which the functions $g_m(y)$ satisfy these boundary conditions. When one of the y boundaries of a rectangular plate is free, however, it is much more difficult to obtain functions which satisfy the conditions at that edge.

As with Galerkin's method, the difficulties are further increased when the plate has free edges which are not parallel to the x axis. It would be impractical to try and obtain the natural frequencies of a cantilevered, trapezoidal plate using Kantorovich's method as it would be expressed, however, in terms of points inside the plate from the finite-difference expressions for the boundary conditions (Williams, 1960, p. 139). Equation 8.14 for the points on or adjacent to the boundary is adjusted accordingly.

If the finite-difference method is used to solve equation 3.1 for trapezoidal or triangular shaped plates, it is convenient to adjust the width of the grid to ensure that the points lie on the sloping edges of the plate (Fig. 8.4). That procedure can lead to a large number of equations. The use of a coarser net to reduce the number of equations would result in the boundaries of the plate cutting across the lines of the grid (Fig. 8.5). The finite-difference expressions for points adjacent to the boundary would have to be adjusted. (Allen, 1954, p. 118).

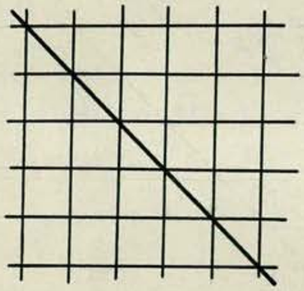


Fig. 8.4

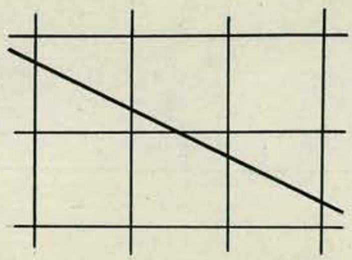


Fig. 8.5

The natural frequencies of the plate may be obtained from the finite-difference expressions by equating the determinant of the coefficients of W to zero. A solution would also be obtained by relaxation methods (Allen, 1954, p. 167). To obtain accurate results, however, equation 3.1 must be replaced at a large number of points. If the former method is used to obtain a solution the use of a digital computer is essential, and if the relaxation method is used the process is slow and tedious.

Appendix No. 5: Calculation of the natural frequencies of the family with one nodal line perpendicular to the fixed edge of cantilevered symmetrical plates

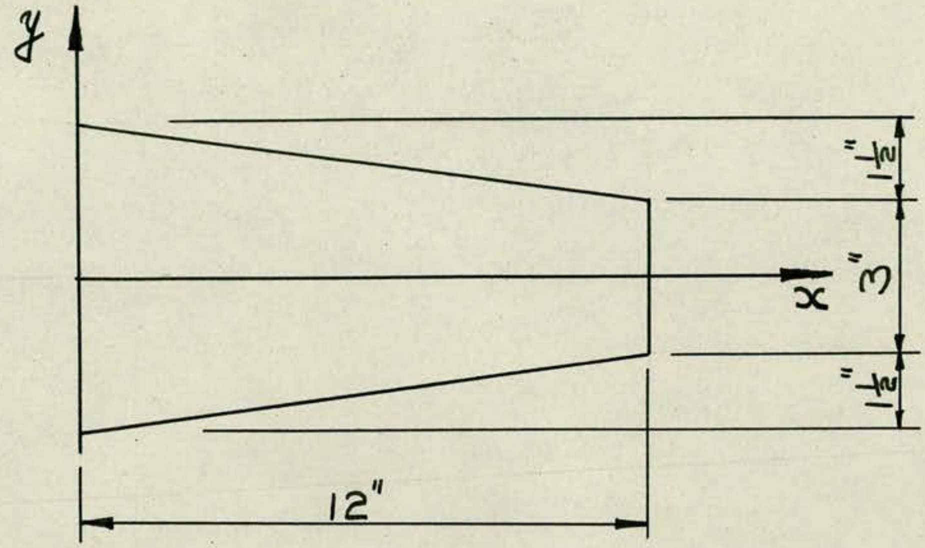


Fig. 8.6

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family m/1 of the symmetrical trapezoidal plate which is shown in Fig. 8.6, the deflected form of the vibrating plate was assumed to be,

$$W = (a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6 + a_6x^7)y + (a_7x^2 + a_8x^3 + a_9x^4 + a_{10}x^5 + a_{11}x^6 + a_{12}x^7)y^3 \qquad 8.15$$

Substituting equation 8.15 in the expressions for the maximum potential energy and the maximum kinetic energy of the vibrating plate and performing the necessary integrations and differentiations leads to twelve simultaneous algebraic equations of the type

$$\left.\begin{array}{l} (c_{11} - \lambda k_{11})a_1 + --- + (c_{112} - \lambda k_{112})a_{12} = 0 \\ (c_{21} - \lambda k_{21})a_1 + --- + (c_{212} - \lambda k_{212})a_{12} = 0 \\ ------------ \\ (c_{121} - \lambda k_{121})a_1 + -- + (c_{1212} - \lambda k_{1212})a_{12} = 0 \end{array}\right\} \qquad 8.16$$

Table 41: Values of non-dimensional frequency factor λ, which were obtained from Ferranti Pegasus Computer when the natural frequencies were calculated using equation 8.15

	Value of λ
1st root	-8.810×10^2
2nd root	6.629×10^3
3rd root	9.904×10^3
4th root	1.854×10^4
5th root	-5.009×10^4
6th root	-2.515×10^4
7th root	2.803×10^5
8th root	-3.964×10^5
9th root	-4.493×10^5
10th root	-7.983×10^5
11th root	-1.618×10^6
12th root	-3.536×10^6

Appendix No. 6: Calculation of the natural frequencies of the family with two nodal lines perpendicular to the fixed edge of cantilevered, symmetrical plates

Using the Rayleigh-Ritz method to calculate the natural frequencies of the family with two nodal lines perpendicular to the fixed edge of the cantilevered symmetrical plates investigated experimentally, the deflected form of the vibrating plate was assumed, in turn, to be

$$W = a_1x^2 + a_2x^3 + a_3x^2y^2 \qquad 8.20$$

$$W = a_1x^2 + a_2x^2y^2 \qquad 8.21$$

$$W = a_1x^2 + a_2x^2y^2 + a_3x^3y^2 \qquad 8.22$$

$$W = a_1x^2 + a_2x^2y^2 + a_3x^3y^2 + a_4x^4y^2 \qquad 8.23$$

With the deflected form of the vibrating plate assumed to be equations 8.20, 8.21, and 8.22 the natural frequency of the first mode of the family of the cantilevered rectangular plate was calculated. Using equation 8.23 to represent the deflected form of the vibrating plate, the natural frequency of the first mode of the family of the cantilevered rectangular and the cantilevered isosceles triangular plates were calculated. The calculated and experimental results are shown in Table 42.

Table 42: Experimental and Calculated Natural Frequencies of mode 1/2 of Cantilevered, Rectangular and Isosceles Triangular Plates

	Rectangular Plate	Triangular Plate
Deflected Form	Natural Frequencies (c.p.s.)	
$W = a_1x^2 + a_2x^3 + a_3x^2y^2$	1998	
$W = a_1x^2 + a_2x^2y^2$	1990	
$W = a_1x^2 + a_2x^2y^2 + a_3x^3y^2$	1730	
$W = a_1x^2 + a_2x^2y^2 + a_3x^3y^2 + a_4x^4y^2$	1543	3300
Experimental Value	1561	3785

The results show that when equations 8.20, 8.21 and 8.22 are used to calculate the natural frequency of mode 1/2 of the cantilevered, rectangular plate the value of the calculated frequency is 28%, 27.5% and 11.9% greater than the experimental value. When equation 8.23 is used to calculate the natural frequency of mode 1/2 of the cantilevered, rectangular plate, the value of the calculated frequency is 1.2% less than the experimental value. Since the calculated frequency, which was obtained using equation 8.23, is much better than that which was obtained using either equations 8.20, 8.21 or 8.22, it was decided to investigate the possibility of using equation 8.23 to calculate the natural frequency of mode 1/2 of the cantilevered, isosceles triangular plate. It can be seen, however, that the value of the calculated frequency of mode 1/2 of the cantilevered isosceles triangular plate is 12.8% less than the experimental value. If the Rayleigh-Ritz method is to prove successful for the calculation of the natural frequencies of the family m/2 of cantilevered rectangular, symmetrical, trapezoidal and isosceles triangular plates some other combination of algebraic functions will have to be used to represent the deflected form of the vibrating plate. No other deflected forms were investigated for the calculation of the natural frequencies of the family m/2 of cantilevered, rectangular, symmetrical trapezoidal and isosceles triangular shaped plates. A suitable deflected form is, therefore, at present unknown to the author. The results are included, however, to complete the record of the work which was done during the research programme.

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